

SIMPLY CONNECTED SURFACES OF GENERAL TYPE IN POSITIVE CHARACTERISTIC VIA DEFORMATION THEORY

YONGNAM LEE AND NOBORU NAKAYAMA

ABSTRACT. Algebraically simply connected surfaces of general type with $p_g = q = 0$ and $1 \leq K^2 \leq 4$ in positive characteristic (with one exception in $K^2 = 4$) are presented by using a \mathbb{Q} -Gorenstein smoothing of two-dimensional toric singularities, a generalization of Lee–Park’s construction [33] to the positive characteristic case, and Grothendieck’s specialization theorem for the fundamental group.

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1. INTRODUCTION

One of the interesting problems in the classification of algebraic surfaces is to find a new family of simply connected surfaces of general type with geometric genus $p_g = 0$. Surfaces with $p_g = 0$ are interesting in view of Castelnuovo’s criterion: *An irrational surface with irregularity $q = 0$ must have bigenus $P_2 \geq 1$.* Simply connected surfaces of general type with $p_g = 0$ are little known and the classification is still open.

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When a surface is defined over the field \mathbb{C} of complex numbers, the only known simply connected minimal surfaces of general type with $p_g = 0$ were Barlow surfaces [7] until 2006. The canonical divisor of Barlow surfaces satisfies $K^2 = 1$. Recently, the first named author and J. Park [33] have constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by using a \mathbb{Q} -Gorenstein smoothing and Milnor fiber of a smoothing (or rational blow-down surgery). When a surface is defined over a field of positive characteristic, the existence of algebraically simply connected minimal surface of general type with $p_g = 0$ is known only for some special characteristics. Lang [28] has constructed surfaces of general type with $p_g = 0$ and $K^2 = 1$ in characteristic 5. Ekedahl [13] has given examples of surfaces of general type with $p_g = 0$ and with $1 \leq K^2 \leq 9$ in characteristic two. The inequality $1 \leq K_S^2 \leq 9$ holds by results of Shepherd-Barron [50], [51] when S is a minimal surface of general type with $p_g = 0$. This is shown in Liedtke's lecture notes [35] on algebraic surfaces in positive characteristic.

We shall construct such a surface of general type defined over an algebraically closed field of any characteristic applying the Lee–Park construction given in [33]. The following is our main result:

Main Theorem. *For any algebraically closed field \mathbb{k} and for any integer $1 \leq K^2 \leq 4$, there exists an algebraically simply connected minimal surface \mathbb{S} of general type over \mathbb{k} with $p_g(\mathbb{S}) = q(\mathbb{S}) = \dim H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = 0$ and $K_{\mathbb{S}}^2 = K^2$ except $(\text{char}(\mathbb{k}), K^2) = (2, 4)$, where $\Theta_{\mathbb{S}/\mathbb{k}}$ denotes the tangent sheaf. Moreover, one can find such a surface with ample canonical divisor when $1 \leq K^2 \leq 4$, except $(\text{char}(\mathbb{k}), K^2) = (2, 1)$, $(2, 2)$, and $(2, 4)$.*

Remark. (1) The surface \mathbb{S} in Main Theorem is liftable to characteristic zero since $H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = H^2(\mathbb{S}, \mathcal{O}_{\mathbb{S}}) = 0$.

- (2) In our method of constructing \mathbb{S} , the bound $K^2 \leq 4$ is necessary (cf. Remark 6.1 below).
- (3) The ampleness property in Main Theorem is known in characteristic zero by [44, Section 2] and the proof of [43, Theorem 4.1] in the case of $K^2 = 3$ and 4.
- (4) We have the exceptional cases of $(\text{char}(\mathbb{k}), K^2)$ by the lack of certain examples of rational surfaces X in Section 6. This is a technical reason, and the existence of such a surface \mathbb{S} in the exceptional cases is an open problem.

The Lee–Park construction is as follows in the case of $K^2 = 2$ (cf. [33, Section 3]): First, we consider a special pencil of cubics in \mathbb{P}^2 and blow up many times to get a projective surface M (\tilde{Z} in [33]) which contains a disjoint union of five linear chains of smooth rational curves representing the resolution graphs of special quotient singularities called of class T (cf. Definition 3.2 below). Then, we contract these linear chains of rational curves from the surface M to produce a projective surface X with five special

quotient singularities of class T and with $K_X^2 = 2$. We can prove the existence of a global \mathbb{Q} -Gorenstein smoothing of the singular surface X (cf. [33, Theorem 4]), in which a general fiber X_t of the \mathbb{Q} -Gorenstein smoothing is a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ (cf. [33, Proposition 8]). The method of Lee–Park construction works to other types of rational elliptic surfaces, which are used to construct a simply connected minimal surface of general type with $p_g = 0$ and with $K^2 = 1, 3$, or 4 ([33, Section 7], [42], [43]). We shall show that the Lee–Park construction of singular surfaces also works in positive characteristic, but several key parts in the proof to show the existence of a global \mathbb{Q} -Gorenstein smoothing should be modified.

Over the field \mathbb{C} of complex numbers, the existence of a local \mathbb{Q} -Gorenstein smoothing of a singularity of class T is given by the index-one cover (cf. [37, Proposition 5.9], [26, Proposition 3.10]). The key idea in [33] to show the vanishing of the obstruction space for a global \mathbb{Q} -Gorenstein smoothing is to use the lifting property of derivations of normal surface to its minimal resolutions, the tautness of the quotient singularities, and the special configurations of resolution graphs of singular points.

In characteristic 0, the tautness holds for quotient singularities (cf. [9, Satz 2.10]), i.e., the minimal resolution graph of a quotient singularity determines the type of singularity. It is known in characteristic 0 that the tautness is equivalent to $H^1(\Theta_D) = 0$ for any “sufficiently large” effective divisor D supported on the exceptional divisors on its minimal resolution (cf. [30, Theorem 3.10], [31, Section 2]). However, tautness does not hold in characteristic $p > 0$ in general. Indeed, we have some examples of rational double points when $p = 2, 3$, and 5 , by Artin’s classification in [6]. The lifting property of derivations from a normal surface to its minimal resolution exists in characteristic 0 ([10, Proposition 1.2]), but this is not true in characteristic $p > 0$ (cf. [5, Example, pp. 345] and Proposition 2.11.(4) below). In [53], the proof of Theorem C illustrates why the lifting property is guaranteed to hold only in characteristic 0, and also provides sufficient conditions for its truth in characteristic $p > 0$.

However, in our constructions, we have only two-dimensional toric singularities by contracting linear chains of smooth rational curves. In Section 2 below, we prove that the singularity obtained by contracting a linear chain of smooth rational curves is a two-dimensional toric singularity and it is taut (cf. Theorem 2.7 below). Moreover, it turns out that the lifting property of derivations mentioned above is not so important for proving the vanishing of the obstruction space (cf. Corollary 2.12 below).

In Section 3, we introduce the notion of toric surface singularity of class T (cf. Definition 3.2), explain some properties with Tables 1 and 2 of related numbers for some special cases, and construct a so-called \mathbb{Q} -Gorenstein smoothing explicitly by using toric description in Theorem 3.8.

We recall several basics on the deformation theory of schemes in Section 4 including Schlessinger's theory of functors on Artinian rings. As an exercise, using cotangent complexes and obstruction theory, we shall prove in Theorem 4.6 that *if $H^2(X, \Theta_{X/\mathbb{k}}) = 0$ for an algebraic \mathbb{k} -variety X with only isolated singularities, then the morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_X^{(\mathrm{loc})}$ between the global and local the deformation functors of X is smooth in the sense of Schlessinger* (cf. [55, Proposition 6.4]). An algebraization result (Theorem 4.7) is added, which plays an important role when we construct an algebraic deformation. The proof uses Artin's theory of algebraization (cf. [3]).

In Section 5, we shall construct a deformation of a normal projective surface X with toric singularities of class T assuming some extra conditions. As a consequence, we have a so-called \mathbb{Q} -Gorenstein smoothing not only over the base field \mathbb{k} but also over a complete discrete valuation ring with the residue field \mathbb{k} (cf. Theorems 5.2 and 5.4). As a byproduct, in Corollary 5.3, we can give a correct version of the existence of \mathbb{Q} -Gorenstein smoothing of any log del Pezzo surfaces of index two in positive characteristic (cf. [40, Theorem 5.16]). By the smoothing over the discrete valuation ring and the Grothendieck specialization theorem (cf. [SGA1, Exp. X, Corollaire 2.4, Théorème 3.8]), the algebraic simply connectedness of the smooth fiber is reduced to that of a smooth fiber of a \mathbb{Q} -Gorenstein smoothing of a reduction of X of our construction to the complex number field \mathbb{C} . Note that the simply connectedness in case $\mathbb{k} = \mathbb{C}$ for the known Lee–Park constructions has been proved in [33], [42], and [43] by using Milnor fiber (or rational blow-down) and by applying van-Kampen's theorem on the minimal resolution of X .

Our plan of the proof of the Main Theorem is as follows: We first construct a normal rational surface X with only toric singularities of class T satisfying extra conditions by Lee–Park's method. This X is constructed from a suitable cubic pencil Φ on \mathbb{P}^2 , a special construction of a birational morphism $M \rightarrow \mathbb{P}^2$, and a blowdown $M \rightarrow X$ of a union of linear chains of rational curves defining toric singularities of class T. Second, we apply the results in Section 5 to X and obtain a so-called \mathbb{Q} -Gorenstein smoothing of X , in which a general smooth fiber X_t is an expected surface satisfying the required conditions in Main Theorem.

In Section 6, we shall explain the outline of our proof giving sufficient conditions for the data needed in constructing X . During the discussion, from the conditions, we shall prove some key results such as the vanishing $H^2(X, \Theta_{X/\mathbb{k}}) = 0$, and the nef and bigness of K_X .

In Section 7, we shall give eight examples working in our proof. Most examples are taken from [33], [42], and [43], but Examples 7.6–7.8 are new which are presented by Heesang Park. These new examples are needed because of some problems in small characteristics. For example, rational surfaces admitting a minimal elliptic fibration with a

configuration type (I_8, I_1, I_1, I_2) of singular fibers are used in [42] and [43], but in characteristic 2, such a configuration does not exist [29]. Moreover, the vanishing $H^2(X, \Theta_{X/\mathbb{k}}) = 0$ does not hold in small characteristic in some cases. By checking the conditions of Section 6 for all the examples, we finally complete the proof of the Main Theorem.

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Notation and conventions. In this article, we fix an algebraically closed field \mathbb{k} of characteristic $p \geq 0$.

- An *algebraic scheme* over a field \mathbb{K} means a \mathbb{K} -scheme of finite type. This is called also an *algebraic \mathbb{K} -scheme*. An *algebraic variety* over a field \mathbb{K} (or an *algebraic \mathbb{K} -variety*) is an integral separated algebraic scheme over \mathbb{K} .

- For an algebraic \mathbb{k} -scheme X , $\Omega_{X/\mathbb{k}}^1$ denotes the sheaf of one-forms and $\Theta_{X/\mathbb{k}}$ denotes the tangent sheaf, i.e., $\Theta_{X/\mathbb{k}} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/\mathbb{k}}^1, \mathcal{O}_X)$. For a non-singular algebraic \mathbb{k} -variety X and a normal crossing divisor B , $\Omega_{X/\mathbb{k}}^1(\log B)$ denotes the sheaf of logarithmic one-forms with poles along only B . Its dual $\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/\mathbb{k}}^1(\log B), \mathcal{O}_X)$ is denoted by $\Theta_{X/\mathbb{k}}(-\log B)$, which is identified with the sheaf of derivations $\delta \in \Theta_{X/\mathbb{k}}$ such that $\delta(\mathcal{O}_X(-B)) \subset \mathcal{O}_X(-B)$.

- For a normal integral separated scheme X and for a (Weil) divisor D , the reflexive sheaf $\mathcal{O}_X(D)$ of rank one is by definition the subsheaf of the sheaf of rational functions determined by the following property: A non-zero rational function φ is contained in $H^0(\mathcal{U}, \mathcal{O}_X(D))$ for a non-empty open subset \mathcal{U} if and only if $\text{div}(\varphi)|_{\mathcal{U}} + D|_{\mathcal{U}} \geq 0$ for the principal divisor $\text{div}(\varphi)|_{\mathcal{U}}$ on \mathcal{U} .

- The maximal ideal of a local ring A is denoted by \mathfrak{m}_A .

- A *geometric point* t of a scheme X is by definition a morphism $t: \text{Spec } \mathbb{k}(t) \rightarrow X$ for an algebraically closed field $\mathbb{k}(t)$. For any scheme Z over X , the geometric fiber Z_t over the geometric point t is defined to be the fiber product $Z \times_{X, t} \text{Spec } \mathbb{k}(t)$.

- An *étale neighborhood* of a pair (X, x) of a scheme X and a point x is by definition a pair (X', x') of a scheme X' étale over X and a point x' lying over x such that the induced homomorphism between the residue fields at x and x' is isomorphic (cf. [2, Section 2]).

• In this article, the (étale) fundamental group of a scheme X is called the *algebraic fundamental group* and is denoted by $\pi_1^{\text{alg}}(X)$ to avoid confusion with the topological fundamental group of a complex algebraic scheme (cf. [SGA1, Exp. XII, Corollaire 5.2]).

2. LINEAR CHAINS OF RATIONAL CURVES AS EXCEPTIONAL LOCI

The main purpose of this section is to prove that any two-dimensional singularity having a linear chain of smooth rational curves as the exceptional locus of the minimal resolution is “toric.” In other words, we prove the tautness (cf. [31]) of such singularities. In the case of characteristic zero, the singularity is just the cyclic quotient singularity of type $\frac{1}{n}(1, q)$ for some integers $n > q > 0$ with $\gcd(n, q) = 1$, and the tautness is known by [9, Satz 2.10].

In this section, let us fix a Noetherian ring Λ and positive integers n, q such that $n > q$ and $\gcd(n, q) = 1$. We define integers b_1, \dots, b_l greater than one by the continued fraction:

$$n/q = [b_1, \dots, b_l] := b_1 - \frac{1}{b_2 - \frac{1}{\ddots - \frac{1}{b_l}}}$$

and introduce the following two conditions:

Definition 2.1 (Conditions $C(n, q)$ and $C(n, q)'$). Let Y be an affine algebraic flat Λ -scheme and let Σ be a closed subscheme such that

- every geometric fiber of $Y \rightarrow \text{Spec } \Lambda$ is a normal surface,
- Σ is a section of $Y \rightarrow \text{Spec } \Lambda$, and
- $Y \setminus \Sigma$ is smooth over $\text{Spec } \Lambda$.

The pair (Y, Σ) is said to satisfy *condition $C(n, q)$* with a proper surjective morphism $\mu: M \rightarrow Y$ if the following conditions are satisfied:

- (1) $M \rightarrow Y \rightarrow \text{Spec } \Lambda$ is smooth and μ is an isomorphism over $Y \setminus \Sigma$.
- (2) $\mu^{-1}(\Sigma)$ is a divisor $\sum_{i=1}^l E_i$ such that:
 - (a) Any E_i is a Cartier divisor with $E_i \simeq \mathbb{P}_\Lambda^1$ and $\mathcal{O}_{E_i}(-E_i) \simeq \mathcal{O}(b_i)$.
 - (b) $E_i \cap E_j = \emptyset$ if $|i - j| > 1$.
 - (c) The scheme-theoretic intersection $\Sigma_i := E_{i-1} \cap E_i$ is a section over $\text{Spec } \Lambda$ for all $2 \leq i \leq l$.

If, in addition, there exist divisors B_1, B_2 on Y satisfying the conditions below, then (Y, B_1, B_2) is said to satisfy *condition $C(n, q)'$* :

- (3) The set-theoretic intersection $B_1 \cap B_2$ is Σ .

- (4) There is a relative Cartier divisor E_0 (resp. E_{l+1}) with respect to $M \rightarrow \text{Spec } \Lambda$ such that:
- (a) $E_0 \cap E_i = \emptyset$ and $E_j \cap E_{l+1} = \emptyset$ for all $i > 1$ and $j < l$.
 - (b) The scheme-theoretic intersections $\Sigma_1 := E_0 \cap E_1$ and $\Sigma_{l+1} := E_l \cap E_{l+1}$ are sections of $M \rightarrow \text{Spec } \Lambda$.
 - (c) B_1 (resp. B_2) is the image of E_{l+1} (resp. E_0) by $\mu: M \rightarrow Y$.

Remark 2.1.1. In the situation above, the “dual graph” of $\sum_{i=0}^{l+1} E_i$ is written as

$$(II-1) \quad \begin{array}{ccccccc} E_0 & & E_1 & & E_2 & & & & E_l & & E_{l+1} \\ \bullet & \text{---} & \circ & \text{---} & \circ & \text{---} & \text{---} & \text{---} & \circ & \text{---} & \bullet \end{array}$$

where the ends are distinguished, since these are not μ -exceptional.

Remark 2.1.2. If Λ is an algebraically closed field, then the condition $C(n, q)$ means that Y is a normal affine surface with a singular point Σ in which the exceptional locus of the minimal resolution is a linear chain of smooth rational curves. Moreover, in this case the condition $C(n, q)'$ means that B_1 and B_2 are prime divisors on Y with $B_1 \cap B_2 = \Sigma$ in which the dual graph of $\mu^{-1}(B_1 \cup B_2)$ is also a linear chain and the end components correspond to the proper transforms of B_1 and B_2 .

In this section we will do the following:

- Giving an explicit construction of a toric Λ -scheme V with two boundary divisors D_1, D_2 such that (V, D_1, D_2) satisfies $C(n, q)'$.
- Showing that any (Y, Σ) satisfying $C(n, q)$ is “étale equivalent to” V along Σ when Λ is a field (cf. Theorem 2.6).
- Showing that any (Y, B_1, B_2) satisfying $C(n, q)'$ is “étale over” (V, D_1, D_2) when Λ is a local ring (cf. Theorem 2.7).
- As applications of Theorems 2.6 and 2.7, giving some local properties of the surface singularity which has a linear chain of smooth rational curves as the exceptional locus of the minimal resolution.

Remark. When Λ is an algebraically closed field, Theorem 2.6 corresponds to the “tautness” (cf. [31]) for two-dimensional normal singularities having linear chains of smooth rational curves as the exceptional locus of the minimal resolution. In positive characteristic, the tautness for linear chains of smooth rational curves seems to be well-known, but the authors could not find any reference. Recently, Hara has found another proof of Theorem 2.6 (cf. Case (1) in the proof of [17, Theorem 2.1]) using an argument in [8].

We begin with constructing V by applying the theory of toric varieties or of torus embeddings. We refer the reader to [12], [24], [11], [41], etc. for more details of the theory.

Let \mathbf{N}_0 be a free abelian group of rank two with a basis (e_1, e_2) , i.e., $\mathbf{N}_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2$. For the fixed integers n and q above, we set

$$v := (1/n)(e_1 + qe_2) \in \mathbf{N}_{0,\mathbb{Q}} = \mathbf{N}_0 \otimes \mathbb{Q} \quad \text{and} \quad \mathbf{N} := \mathbf{N}_0 + \mathbb{Z}v \subset \mathbf{N}_{0,\mathbb{Q}}.$$

Now, we define V to be the affine toric Λ -scheme $\mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$ associated with \mathbf{N} and with the cone $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 \subset \mathbf{N}_{\mathbb{R}}$. More precisely, $V := \text{Spec } \Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}]$ for the semi-group ring $\Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}]$, where $\mathbf{M} = \text{Hom}(\mathbf{N}, \mathbb{Z})$ and $\boldsymbol{\sigma}^\vee = \{m \in \mathbf{M}_{\mathbb{R}} \mid m \geq 0 \text{ on } \boldsymbol{\sigma}\}$. Note that $\Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}_0]$ is a polynomial algebra $\Lambda[\mathbf{x}_1, \mathbf{x}_2]$ of two variables, where $\mathbf{M}_0 = \text{Hom}(\mathbf{N}_0, \mathbb{Z})$ and that the subalgebra $\Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}]$ is generated by monomials $\mathbf{x}_1^{m_1}\mathbf{x}_2^{m_2}$ for integers m_1, m_2 such that $m_1 \geq 0, m_2 \geq 0$, and $m_1 + qm_2 \equiv 0 \pmod{n}$. For the group ring $\Lambda[\mathbf{M}]$, the affine Λ -scheme $\mathbb{T}_{\mathbf{N}} = \text{Spec } \Lambda[\mathbf{M}]$ is a group scheme isomorphic to $\mathbb{G}_{\mathbf{m},\Lambda}^2$, the algebraic torus of relative dimension two. Since $\boldsymbol{\sigma}^\vee \cap \mathbf{M} \subset \mathbf{M}$, V contains $\mathbb{T}_{\mathbf{N}}(\{0\}) = \mathbb{T}_{\mathbf{N}}$ as an open subset. Moreover, the standard action of $\mathbb{T}_{\mathbf{N}}$ on $\mathbb{T}_{\mathbf{N}}$ extends to V , compatibly with the open immersion $\mathbb{T}_{\mathbf{N}} \subset V$.

Definition 2.2. The Λ -scheme V is said to be the *toric Λ -scheme of type (n, q)* .

For an element $m \in \mathbf{M}$, let $\mathbf{e}(m)$ be the element in the group algebra $\Lambda[\mathbf{M}]$ corresponding to m . Then, $\mathbf{e}(m)$ can be regarded as a rational function on V , and it is regular when $m \in \boldsymbol{\sigma}^\vee \cap \mathbf{M}$. The cone $\boldsymbol{\sigma}$ has four faces: $\boldsymbol{\sigma}$ itself, two rays $\mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2$, and the zero-dimensional cone $\{0\}$. For such a face $\boldsymbol{\tau}$, let $\boldsymbol{\tau}^\perp$ be the vector subspace $\{m \in \mathbf{M}_{\mathbb{R}} \mid m(\boldsymbol{\tau}) = 0\}$ and consider the ring homomorphism $\Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}] \rightarrow \Lambda[\boldsymbol{\tau}^\perp \cap \mathbf{M}]$ defined by

$$\mathbf{e}(m) \mapsto \begin{cases} \mathbf{e}(m), & \text{if } m \in \boldsymbol{\tau}^\perp; \\ 0, & \text{otherwise.} \end{cases}$$

The kernel of the ring homomorphism defines a closed subscheme $Z(\boldsymbol{\tau})$, which is the closure of an orbit of $\mathbb{T}_{\mathbf{N}}$ on V . We set $D_i := Z(\mathbb{R}_{\geq 0}e_i)$ for $i = 1, 2$, and set $\Sigma := Z(\boldsymbol{\sigma})$. Then, the following properties are easily shown:

- D_1 and D_2 are prime divisors on V when Λ is integral.
- Σ is a section of V over $\text{Spec } \Lambda$.
- Σ is the scheme-theoretic intersection $D_1 \cap D_2$ (cf. Remark 2.2.1 below).
- $V \setminus (D_1 \cup D_2)$ is the “open torus” $\mathbb{T}_{\mathbf{N}}(\{0\})$.
- For $i = 1, 2$, $D_i \simeq \mathbb{A}_{\Lambda}^1$ and $D_i \setminus \Sigma \simeq \mathbb{G}_{\mathbf{m},\Lambda}$.
- $\mathbb{T}_{\mathbf{N}}(\{0\})$, $D_1 \setminus \Sigma$, $D_2 \setminus \Sigma$, and Σ are the orbits of $\mathbb{T}_{\mathbf{N}}$ in V .

Note also that, for $i = 1, 2$, the order of zeros (or the minus of the order of poles) of the rational function $\mathbf{e}(m)$ for $m \in \mathbf{M}$ along the prime divisor D_i equals $m(e_i)$ (cf. [24, Chapter I, Theorem 9], [11, §5], [41, Section 2.1]). Hence, the principal divisor $\text{div}(\mathbf{e}(m))$

has the following expression:

$$(II-2) \quad \operatorname{div}(\mathbf{e}(m)) = m(e_1)D_1 + m(e_2)D_2.$$

Remark 2.2.1. Using (II-2), we can prove that Σ is just the scheme-theoretic intersection of D_1 and D_2 as follows. For $i = 1, 2$, the defining ideal I_i of D_i is generated by $\mathbf{e}(m)$ for $m \in \sigma^\vee \cap \mathbf{M}$ with $m(e_i) > 0$ by (II-2). On the other hand, the defining ideal I at Σ is generated by $\mathbf{e}(m)$ for $m \in \sigma^\vee \cap \mathbf{M} \setminus \{0\}$. Thus, $I = I_1 + I_2$, i.e., Σ is the scheme-theoretic intersection $D_1 \cap D_2$.

Remark 2.2.2. If Λ' is a Λ -algebra, then $V \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \Lambda'$ is also the toric Λ' -scheme of type (n, q) . If Λ is an algebraically closed field, then V is an affine normal surface with unique singular point Σ .

In order to show that (V, D_1, D_2) satisfies the condition $C(n, q)'$, we shall construct the toric “minimal resolution of singularities” $\nu: U \rightarrow V$ which is essentially the same as the well-known Jung–Hirzebruch resolution (cf. [23], [19]). The following lemma follows from the property $n/q = [b_1, \dots, b_l]$: the proof is left to the reader.

Lemma 2.3. *There exist vectors $v_0, v_1, \dots, v_{l+1} \in \mathbf{N} \cap \sigma$ satisfying the following conditions:*

(1) *Let p_i, q_i be integers determined by $v_i = (1/n)(p_i e_1 + q_i e_2)$. Then,*

$$p_0 = 0 < p_1 = 1 < \dots < p_{l+1} = n \quad \text{and} \quad q_0 = n > q_1 = q > \dots > q_l = 1 > q_{l+1} = 0.$$

(2) *$v_{i-1} + v_{i+1} = b_i v_i$ for $1 \leq i \leq l$.*

(3) *$\mathbf{N} = \mathbb{Z}v_{i-1} + \mathbb{Z}v_i$ for $1 \leq i \leq l$.*

(4) *Let (h_1, h_2) be the basis of $\mathbf{M}_0 = \operatorname{Hom}(\mathbf{N}_0, \mathbb{Z})$ dual to (e_1, e_2) , i.e., $h_i(e_j) = \delta_{i,j}$.*

Then, for all $1 \leq i \leq l$,

$$(-q_i h_1 + p_i h_2, q_{i-1} h_1 - p_{i-1} h_2)$$

is the basis of \mathbf{M} dual to (v_{i-1}, v_i) .

(5) *$p_i q_j \equiv p_j q_i \pmod{n}$ for all $1 \leq i, j \leq l$.*

Note that $v_0 = e_2$, $v_1 = v$, and $v_{l+1} = e_1$. Note also that the last assertion (5) is derived from (4) since it is equivalent to: $(-q_i h_1 + p_i h_2)(v_j) \in \mathbb{Z}$ for all $1 \leq i, j \leq l$.

The cones $\sigma_i := \mathbb{R}_{\geq 0}v_{i-1} + \mathbb{R}_{\geq 0}v_i$ for $1 \leq i \leq l$ and the rays $\mathbb{R}_{\geq 0}v_j$ for $0 \leq j \leq l+1$ together with the zero cone $\{0\}$ form a non-singular fan Δ of \mathbf{N} such that the support $|\Delta|$ coincides with σ (cf. [12, Section 4.2], [24, Chapter I, §2], [11, §5], [41, Chapter 1]). Let U be the toric Λ -scheme $\mathbb{T}_{\mathbf{N}}(\Delta)$ associated with the fan Δ . Then, we have a canonical proper surjective morphism $\nu: U \rightarrow V$ which is an isomorphism at least on the open torus

$\mathbb{T}_{\mathbf{N}}(\{0\})$. Now, U is a union of the smooth affine toric Λ -schemes $U_i = \operatorname{Spec} \Lambda[\boldsymbol{\sigma}_i^\vee \cap \mathbf{M}] \simeq \mathbb{A}_\Lambda^2$. In fact, by Lemma 2.3, we see that $\Lambda[\boldsymbol{\sigma}_i^\vee \cap \mathbf{M}]$ is a polynomial Λ -algebra of two variables generated by

$$\xi_i := \mathbf{e}(-q_i h_1 + p_i h_2) = \mathbf{x}_1^{-q_i} \mathbf{x}_2^{p_i} \quad \text{and} \quad \eta_i := \mathbf{e}(q_{i-1} h_1 - p_{i-1} h_2) = \mathbf{x}_1^{q_{i-1}} \mathbf{x}_2^{-p_{i-1}}.$$

Here, $U_i \cap U_j$ equals the open torus $\mathbb{T}_{\mathbf{N}}(\{0\}) \simeq \mathbb{T}_{\mathbf{N}}$ of U if $|i - j| > 1$, since $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}_j = \{0\}$. For $1 \leq i \leq l$, the intersection $U_i^* := U_i \cap U_{i+1}$ is isomorphic to the toric Λ -scheme $\mathbb{T}_{\mathbf{N}}(\mathbb{R}_{\geq 0} v_i) \simeq \mathbb{A}_\Lambda^1 \times_{\operatorname{Spec} \Lambda} \mathbb{G}_{\mathbf{m}, \Lambda}$, since $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}_{i+1} = \mathbb{R}_{\geq 0} v_i$. For $0 \leq i \leq l + 1$, let G_i be the $\mathbb{T}_{\mathbf{N}}$ -invariant prime divisor on $U = \mathbb{T}_{\mathbf{N}}(\Delta)$ corresponding to the ray $\mathbb{R}_{\geq 0} v_i$. In particular, G_0 (resp. G_{l+1}) is the proper transform of D_2 (resp. D_1).

Lemma 2.4. *The triplet (V, D_1, D_2) satisfies the condition $C(n, q)'$ with the proper morphism $\nu: U \rightarrow V$.*

Proof. For any $m \in \mathbf{M}$, we have

$$(II-3) \quad \operatorname{div}(\mathbf{e}(m)) = \sum_{i=0}^{l+1} m(v_i) G_i$$

as in (II-2), where $\mathbf{e}(m)$ is regarded as a rational function on U . Now, $G_i|_{U_j} = 0$ for $j < i$ and $j > i + 1$, since $v_i \notin \boldsymbol{\sigma}_j$. Furthermore, we have:

$$(II-4) \quad G_i|_{U_i} = \operatorname{div}(\eta_i)|_{U_i} \quad \text{and} \quad G_i|_{U_{i+1}} = \operatorname{div}(\xi_{i+1})|_{U_{i+1}},$$

by the calculation

$$\begin{aligned} \operatorname{div}(\xi_i)|_{U_i} &= \sum_{j=i-1}^i (-q_i h_1 + p_i h_2)(v_j) G_j|_{U_i} = G_{i-1}|_{U_i}, \\ \operatorname{div}(\eta_i)|_{U_i} &= \sum_{j=i-1}^i (q_{i-1} h_1 - p_{i-1} h_2)(v_j) G_j|_{U_i} = G_i|_{U_i}, \end{aligned}$$

using (II-3) and Lemma 2.3. As a consequence,

$$U_i^* = U_i \setminus G_{i-1} = U_i \cap \{\xi_i \neq 0\} \simeq \mathbb{A}_\Lambda^1 \times \mathbb{G}_{\mathbf{m}, \Lambda}.$$

The toric Λ -scheme U is obtained by gluing the affine Λ -planes $U_i \simeq \mathbb{A}_\Lambda^2$ for $1 \leq i \leq l + 1$ with the coordinate systems (ξ_i, η_i) . The transition relation is

$$(II-5) \quad \xi_{i+1}|_{U_i^*} = \xi_i^{b_i} \eta_i|_{U_i^*}, \quad \text{and} \quad \eta_{i+1}|_{U_i^*} = \xi_i^{-1}|_{U_i^*},$$

on $U_i^* = U_i \cap U_{i+1}$. This is possible, since $(-q_{i+1}, p_{i+1}) - (q_{i-1}, -p_{i-1}) = b_i(-q_i, p_i)$ (cf. Lemma 2.3). Furthermore, $G_i \simeq \mathbb{P}_\Lambda^1$ with $\mathcal{O}_{G_i}(G_i) \simeq \mathcal{O}(-b_i)$ for all $1 \leq i \leq l$ by (II-4) and (II-5). On the other hand, G_0 and G_{l+1} are isomorphic to \mathbb{A}_Λ^1 . The scheme-theoretic intersection $\Sigma_i = G_{i-1} \cap G_i$ for $1 \leq i \leq l + 1$ is just the orbit corresponding to the two-dimensional cone $\boldsymbol{\sigma}_i$. Therefore, $\sum_{i=0}^{l+1} G_i$ is a (relative) simple normal crossing divisor (over $\operatorname{Spec} \Lambda$) with the dual graph similar to (II-1), and hence, (V, D_1, D_2) satisfies the condition $C(n, q)'$. \square

Remark 2.4.1. Let p_i and q_i for $0 \leq i \leq l+1$ be the integers defined by (n, q) in Lemma 2.3. If a_1 and a_2 are integers with $a_1 + qa_2 \equiv 0 \pmod n$, then, $p_ia_1 + q_ia_2 \equiv 0 \pmod n$ for all $0 \leq i \leq l+1$, and

$$(II-6) \quad \mu^*(a_1D_1 + a_2D_2) = \sum_{i=0}^{l+1} \frac{p_ia_1 + q_ia_2}{n} G_i.$$

In fact, this is derived from (II-2) and (II-3) applied to $m = a_1h_1 + a_2h_2$. As special cases, we have

$$(II-7) \quad \mu^*(nD_1) = \sum_{i=1}^{l+1} p_i G_i \quad \text{and} \quad \mu^*(nD_2) = \sum_{i=0}^l q_i G_i.$$

We have another proof of (II-6) in Corollary 2.8.3 below.

Next, we consider an arbitrary affine Λ -scheme Y and a closed subscheme Σ satisfying $C(n, q)$. First we shall show:

Lemma 2.5. *Let Y be an affine Λ -scheme and Σ a closed subscheme such that (Y, Σ) satisfies $C(n, q)$ for a proper morphism $\mu: M \rightarrow Y$. Then, $\mathcal{O}_Y \simeq \mu_* \mathcal{O}_M \simeq j_* \mathcal{O}_{Y \setminus \Sigma}$, where $j: Y \setminus \Sigma \hookrightarrow Y$ denotes the open immersion.*

Proof. We take an arbitrary point $y \in \Sigma$ and a point $x \in M$ lying over y . Let $t \in \text{Spec } \Lambda$ be the image of y , and let Y_t and M_t be the fiber over t of $Y \rightarrow \text{Spec } \Lambda$ and $M \rightarrow \text{Spec } \Lambda$, respectively. Then,

$$\begin{aligned} \text{depth } \mathcal{O}_{Y,y} &= \text{depth } \mathcal{O}_{Y_t,y} + \text{depth } \mathcal{O}_{\text{Spec } \Lambda, t} \geq \text{depth } \mathcal{O}_{Y_t,y} = 2, \\ \text{depth } \mathcal{O}_{M,x} &= \text{depth } \mathcal{O}_{M_t,x} + \text{depth } \mathcal{O}_{\text{Spec } \Lambda, t} \geq \text{depth } \mathcal{O}_{M_t,x} \geq 1, \end{aligned}$$

since Y and M are flat over Λ , and since all the fibers of $Y \rightarrow \text{Spec } \Lambda$ and $M \rightarrow \text{Spec } \Lambda$ are normal surfaces. Hence, $\mathcal{O}_Y \rightarrow j_* \mathcal{O}_{Y \setminus \Sigma}$ is an isomorphism and $\mathcal{O}_M \rightarrow j'_* \mathcal{O}_{M \setminus \mu^{-1}(\Sigma)}$ is injective for the open immersion $j': M \setminus \mu^{-1}(\Sigma) \hookrightarrow M$ (cf. [EGA, IV, Théorème (5.10.5), Proposition (5.10.2)] or [SGA2, Exp. III, Proposition 3.3, Corollaire 3.5]). Since $M \setminus \mu^{-1}(\Sigma) \simeq Y \setminus \Sigma$, we have isomorphisms $\mathcal{O}_Y \simeq \mu_* \mathcal{O}_M \simeq j_* \mathcal{O}_{Y \setminus \Sigma}$. \square

The condition $C(n, q)$ characterizes the toric Λ -scheme of type (n, q) up to étale morphism when Λ is a field. Namely, we have:

Theorem 2.6 (tautness). *Assume that Λ is a field. Let (Y, Σ) be a pair satisfying the condition $C(n, q)$ with a proper morphism $\mu: M \rightarrow Y$. Let V be the toric Λ -scheme V of type (n, q) and $\nu: U \rightarrow V$ the minimal resolution constructed as above. Then, there exists an étale neighborhood Y° of Σ in Y with étale morphisms $\tau: Y^\circ \rightarrow V$ and*

$\Phi: M^\circ := M \times_Y Y^\circ \rightarrow U$ which form a commutative Cartesian diagram:

$$\begin{array}{ccc} M^\circ & \xrightarrow{\mu \times_Y Y^\circ} & Y^\circ \\ \Phi \downarrow & & \tau \downarrow \\ U & \xrightarrow{\nu} & V. \end{array}$$

Remark 2.6.1. An étale neighborhood Y° of Σ in Y means an étale morphism $Y^\circ \rightarrow Y$ such that the image contains Σ and that $\Sigma \times_Y Y^\circ \rightarrow \Sigma$ is an isomorphism (cf. Notation and conventions).

Similarly, the condition $C(n, q)'$ characterizes the toric Λ -scheme of type (n, q) up to étale morphism:

Theorem 2.7. *Assume that Λ is a Noetherian local ring. Let (Y, B_1, B_2) be a collection satisfying the condition $C(n, q)'$ with a proper morphism $\mu: M \rightarrow Y$. Let V be the toric Λ -scheme V of type (n, q) and $\nu: U \rightarrow V$ the minimal resolution constructed as above. Then, there exists an open neighborhood Y° of Σ in Y with étale morphisms $\tau: Y^\circ \rightarrow V$ and $\Phi: M^\circ := M \times_Y Y^\circ \rightarrow U$ which form the same commutative Cartesian diagram as in Theorem 2.6.*

Before proving Theorems 2.6 and 2.7, we need some results on the “singularity” of Y along Σ .

Lemma 2.8. *In the situation of Theorem 2.6 or 2.7, if an invertible sheaf \mathcal{L} on M is μ -nef, i.e., $\deg(\mathcal{L}|_C) \geq 0$ for any fiber C of $E_i \simeq \mathbb{P}_\Lambda^1 \rightarrow \Sigma \simeq \text{Spec } \Lambda$ for all $1 \leq i \leq l$, then $H^j(M, \mathcal{L}) = 0$ for all $j > 0$ and \mathcal{L} is generated by global sections. If \mathcal{L} is μ -numerically trivial, i.e., $\deg(\mathcal{L}|_C) = 0$ for any C above, then $\mu_* \mathcal{L}$ is an invertible sheaf on Y and $\mu^*(\mu_* \mathcal{L}) \simeq \mathcal{L}$.*

Proof. For the vanishing $H^j(M, \mathcal{L}) = 0$, it is enough to check only when $j = 1$, since the dimension of the fibers of μ is at most one. Let Z be the divisor $\sum_{i=1}^l E_i$ which is a relative Cartier divisor over $\text{Spec } \Lambda$. Then, $\mathcal{O}_M(-Z)$ is μ -nef by $b_i \geq 2$. It is enough to prove the following two assertions:

- (1) $H^1(Z, \mathcal{L}|_Z) = 0$.
- (2) $\mathcal{L}|_Z$ is generated by global sections.

In fact, $H^1(Z, \mathcal{L}|_Z \otimes \mathcal{O}_Z(-mZ)) = 0$ for any $m \geq 0$ by (1), and it implies the vanishing $H^1(M, \mathcal{O}_{mZ} \otimes \mathcal{L}) = 0$ for all $m \geq 0$ by induction using the exact sequence

$$0 \rightarrow \mathcal{O}_Z(-(m-1)Z) \otimes_{\mathcal{O}_M} \mathcal{L} \rightarrow \mathcal{O}_{mZ} \otimes_{\mathcal{O}_M} \mathcal{L} \rightarrow \mathcal{O}_{(m-1)Z} \otimes_{\mathcal{O}_M} \mathcal{L} \rightarrow 0.$$

Hence, $H^1(M, \mathcal{L}) = 0$, since the module $H^1(M, \mathcal{L})$ is supported on Σ and the formal completion $H^1(M, \mathcal{L})^\wedge$ along Σ is isomorphic to the projective limit $\varprojlim_m H^1(\mathcal{O}_{mZ} \otimes \mathcal{L})$

(cf. [EGA, III, Théorème (4.1.5)]). We also have $H^1(M, \mathcal{O}_M(-Z) \otimes \mathcal{L}) = 0$, since $\mathcal{O}_M(-Z)$ is nef, and as a consequence, the restriction map $H^0(M, \mathcal{L}) \rightarrow H^0(Z, \mathcal{L}|_Z)$ is surjective. Thus, \mathcal{L} is generated by global sections by (2).

For integers $1 \leq a \leq b \leq l$, we set $Z_{a,b} := \sum_{i=a}^b E_i$. We shall show (1) and (2) by proving the following two assertions for any such pair (a, b) of integers:

$$(3) \ H^1(Z_{a,b}, \mathcal{L}|_{Z_{a,b}}) = 0.$$

$$(4) \ \mathcal{L}|_{Z_{a,b}} \text{ is generated by global sections.}$$

We shall prove them by induction on $b - a$. These are true when $a = b$. In fact, $Z_{a,a} = E_a$, $E_a \simeq \mathbb{P}_\Lambda^1$, and $\mathcal{L}|_{E_a} \simeq \mathcal{O}_{\mathbb{P}_\Lambda^1}(d_a)$ for some $d_a \geq 0$. Assume that $b > a$. Then, we have two exact sequences

$$0 \rightarrow \mathcal{O}_{E_a}(-Z_{a+1,b}) \simeq \mathcal{O}_{\mathbb{P}_\Lambda^1}(-1) \rightarrow \mathcal{O}_{Z_{a,b}} \rightarrow \mathcal{O}_{Z_{a+1,b}} \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \mathcal{O}_{E_b}(-Z_{a,b-1}) \simeq \mathcal{O}_{\mathbb{P}_\Lambda^1}(-1) \rightarrow \mathcal{O}_{Z_{a,b}} \rightarrow \mathcal{O}_{Z_{a,b-1}} \rightarrow 0.$$

Since $\mathcal{L}|_{E_a} \simeq \mathcal{O}_{\mathbb{P}_\Lambda^1}(d_a)$ and $\mathcal{L}|_{E_b} \simeq \mathcal{O}_{\mathbb{P}_\Lambda^1}(d_b)$ with $d_a, d_b \geq 0$, we have

$$H^1(E_a, \mathcal{L}|_{E_a} \otimes \mathcal{O}_{E_a}(-Z_{a+1,b})) = H^1(E_b, \mathcal{L}|_{E_b} \otimes \mathcal{O}_{E_b}(-Z_{a,b-1})) = 0.$$

Therefore, we have surjections $H^0(\mathcal{L}|_{Z_{a,b}}) \rightarrow H^0(\mathcal{L}|_{Z_{a+1,b}})$ and $H^0(\mathcal{L}|_{Z_{a,b}}) \rightarrow H^0(\mathcal{L}|_{Z_{a,b-1}})$, and isomorphisms $H^1(\mathcal{L}|_{Z_{a,b}}) \simeq H^1(\mathcal{L}|_{Z_{a+1,b}}) \simeq H^1(\mathcal{L}|_{Z_{a,b-1}})$. Thus, (3) and (4) for (a, b) follow from those for $(a+1, b)$ and for $(a, b-1)$. By induction on $b - a$, (3) and (4) hold for any (a, b) , and hence, (1) and (2) hold. Thus, we are done. \square

Remark. If one knows the vanishing $H^1(M, \mathcal{O}_M) = 0$, then Lemma 2.8 is a special case of [36, Theorem (12.1)], which generalizes [1, Lemma 5] stated over an algebraically closed field.

Corollary 2.8.1. *Let Y be an affine normal algebraic surface defined over an algebraically closed field \mathbb{k} with a unique singular point P . If the exceptional locus for the minimal resolution of a singularity is a linear chain of smooth rational curves, then (Y, P) is a rational singularity.*

Remark 2.8.2. In Corollary 2.8.1, the divisor Z in the proof of Lemma 2.8 is nothing but the fundamental cycle. Thus, the rationality of (Y, P) is also derived from $p_a(Z) = 0$ by [1, Theorem 3].

Corollary 2.8.3. *Let $\mu: M \rightarrow Y$, B_1 , B_2 , and $\{E_i\}_{i=0}^{l+1}$ be as in the situation of Theorem 2.7. Let a_1, a_2 be integers such that $a_1 + qa_2 \equiv 0 \pmod{n}$. Then, $p_i a_1 + q_i a_2 \equiv 0 \pmod{n}$ for all $0 \leq i \leq l+1$ for integers p_i and q_i defined in Lemma 2.3, $a_1 B_1 + a_2 B_2$ is Cartier, and the equality*

$$(II-8) \quad \mu^*(a_1 B_1 + a_2 B_2) = \sum_{i=0}^{l+1} \frac{p_i a_1 + q_i a_2}{n} E_i$$

of Cartier divisors hold on M .

Proof. By Lemma 2.3, we have $p_i a_1 + q_i a_2 \equiv 0 \pmod{n}$, and we see that the right hand side of (II-8) is μ -numerically trivial. Hence by Lemma 2.8, the associated invertible sheaf to the divisor of the right hand side is just the pullback of an invertible sheaf on Y . Thus, (II-8) holds, since $a_1 B_1 + a_2 B_2$ is the push-forward of the right hand side. \square

Proof of Theorems 2.6 and 2.7. First, we shall prove Theorem 2.6 assuming Theorem 2.7 to be true. In Theorem 2.6, Λ is a field, say \mathbb{K} . Thus, Σ is a \mathbb{K} -rational point. Let R be the henselization of the local ring $\mathcal{O}_{Y,\Sigma}$ and let $\mu^\sim: M^\sim \rightarrow \text{Spec } R$ be the base change of $\mu: M \rightarrow Y$ by $\text{Spec } R \rightarrow Y$. Then,

$$\text{Pic}(M^\sim) \rightarrow \text{Pic}\left(\bigcup_{i=1}^l E_i\right)$$

is surjective by [36, Lemma (14.3)] or [EGA, IV, Corollaire (21.9.12)]. Hence, by replacing Y with an étale neighborhood $Y^\circ \rightarrow Y$ of Σ in Y , we may assume that $\text{Pic}(M) \rightarrow \text{Pic}(\bigcup_{i=1}^l E_i)$ is surjective. Then, we can find invertible sheaves $\mathcal{L}, \mathcal{L}'$ on M such that

$$\deg \mathcal{L}|_{E_i} = \deg \mathcal{L}'|_{E_i} = 1, \quad \text{and} \quad \deg \mathcal{L}|_{E_j} = \deg \mathcal{L}'|_{E_j} = 0$$

for $i > 1$ and $j < l$. Applying Lemma 2.8 to \mathcal{L} and \mathcal{L}' , we have two affine prime divisors E_0, E_{l+1} on M such that

- $\mathcal{L} \simeq \mathcal{O}_M(E_0)$ and $\mathcal{L}' \simeq \mathcal{O}_M(E_{l+1})$,
- $E_0 \cap E_1$ and $E_l \cap E_{l+1}$ are sections of $M \rightarrow \text{Spec } \Lambda$, and
- $E_0 \cap \bigcup_{i=2}^{l+1} E_i = E_{l+1} \cap \bigcup_{i=0}^{l-1} E_i = \emptyset$.

Hence, the dual graph of $\sum_{i=0}^{l+1} E_i$ is the same as (II-1). We set $B_1 := \mu_*(E_{l+1})$ and $B_2 := \mu_*(E_0)$. Then, B_1 and B_2 are prime divisors on Y with $B_1 \cap B_2 = \Sigma$, set-theoretically. Furthermore, E_{l+1} and E_0 are the proper transforms of B_1 and B_2 , respectively. Thus, (Y, B_1, B_2) satisfies the condition $C(n, q)'$. Hence, Theorem 2.6 is derived from Theorem 2.7.

In the rest of the proof, we shall prove Theorem 2.7. If a_1 and a_2 are integers with $a_1 + q a_2 \equiv 0 \pmod{n}$, then $a_1 B_1 + a_2 B_2$ is Cartier by Corollary 2.8.3. We may assume that these Cartier divisors $a_1 B_1 + a_2 B_2$ are all linearly equivalent to zero by replacing Y with an open neighborhood of Σ , since Λ is local.

For $0 \leq i \leq l+1$, let ϵ_i be a global section of $\mathcal{O}_M(E_i)$ such that the divisor $(\epsilon_i)_0$ of zeros equals E_i ; in other words, $\epsilon_i: \mathcal{O}_M \rightarrow \mathcal{O}_M(E_i)$ is dual to the natural injection $\mathcal{O}_M(-E_i) \subset \mathcal{O}_M$. Let a_1, a_2 be integers with $a_1 + q a_2 \equiv 0 \pmod{n}$. Then, by (II-8), there is a rational function ϕ_{a_1, a_2} on Y such that

$$\mu^*(\phi_{a_1, a_2}) = \prod_{i=0}^{l+1} \epsilon_i^{(p_i a_1 + q_i a_2)/n}.$$

For $1 \leq i \leq l+1$, let M_i be the complement of $\bigcup_{k \neq i-1, i} E_k$ in M . Then, M_i is a neighborhood of the intersection $E_{i-1} \cap E_i$. We define

$$s_i := \mu^*(\phi_{-q_i, p_i})|_{M_i} \quad \text{and} \quad t_i := \mu^*(\phi_{q_{i-1}, -p_{i-1}})|_{M_i}$$

for $1 \leq i \leq l+1$. By (II-8), we see that s_i and t_i are regular on M_i satisfying

$$\operatorname{div}(s_i)|_{M_i} = E_{i-1}|_{M_i} \quad \text{and} \quad \operatorname{div}(t_i)|_{M_i} = E_i|_{M_i}.$$

In particular, (s_i, t_i) is a local coordinate system of M_i along the intersection $E_{i-1} \cap E_i$.

For $1 \leq i \leq l$, let M_i^* be the intersection $M_i \cap M_{i+1} = M \setminus \bigcup_{k \neq i} E_k$. Then, we have

$$(II-9) \quad s_{i+1}|_{M_i^*} = s_i^{b_i} t_i|_{M_i^*}, \quad t_{i+1}|_{M_i^*} = s_i^{-1}|_{M_i^*},$$

similar to (II-5). For $1 \leq i \leq l+1$, let $\Phi_i: M_i \rightarrow U_i = \operatorname{Spec} \mathbb{k}[\sigma_i^\vee \cap \mathbf{M}] \simeq \mathbb{A}_{\mathbb{k}}^2$ be the morphism defined by

$$\Phi_i^*(\xi_i) = s_i \quad \text{and} \quad \Phi_i^*(\eta_i) = t_i,$$

which is étale along $(E_{i-1} \cup E_i) \cap M_i$. By (II-5) and (II-9), the morphisms Φ_i for $1 \leq i \leq l+1$ are glued to a morphism $\Phi: M = \bigcup M_i \rightarrow U = \bigcup U_i$, which is étale along $\bigcup_{i=0}^{l+1} E_i$, and which induces $\Phi^*(G_i) = E_i$ for all $0 \leq i \leq l+1$. Since $\mu \circ \Phi: M \rightarrow U \rightarrow V$ contracts the divisor $E = \sum_{i=1}^l E_i$ to the section Σ , we have a morphism $\tau: Y \rightarrow V$ such that $\nu \circ \Phi = \tau \circ \mu$, i.e., the diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & Y \\ \Phi \downarrow & & \downarrow \tau \\ U & \xrightarrow{\nu} & V \end{array}$$

is commutative. For the proof of Theorem 2.7, it is enough to prove that τ is étale along Σ . Applying [EGA, III, Théorème (4.1.5)], we have isomorphisms

$$\mathcal{O}_Y^\wedge \simeq \varprojlim_m \mu_*(\mathcal{O}_M/\mathcal{O}_M(-mE)) \quad \text{and} \quad \mathcal{O}_V^\wedge \simeq \varprojlim_m \nu_*(\mathcal{O}_U/\mathcal{O}_U(-mG)),$$

where $E = \sum_{i=1}^l E_i$, $G = \sum_{i=1}^l G_i$, and \mathcal{O}_Y^\wedge (resp. \mathcal{O}_V^\wedge) is the formal completion of $\mathcal{O}_Y \simeq \mu_*\mathcal{O}_M$ (resp. $\mathcal{O}_V \simeq \nu_*\mathcal{O}_U$) along Σ (resp. Σ). On the other hand, Φ induces an isomorphism

$$H^0(U, \mathcal{O}_U/\mathcal{O}_U(-mG)) \simeq H^0(M, \mathcal{O}_M/\mathcal{O}_M(-mE))$$

of Λ -algebras for all $m > 0$, since $\Phi^*(G) = E$ and $\Phi|_E: E \rightarrow G$ is an isomorphism. Thus, τ induces an isomorphism $\mathcal{O}_U^\wedge \simeq \mathcal{O}_Y^\wedge$. Hence, the morphism $\tau_t: Y_t \rightarrow V_t$ between the fibers of V and U over any point $t \in \operatorname{Spec} \Lambda$ induced by τ is étale at the point $\Sigma \cap Y_t$ by [EGA, IV, Théorème (17.6.1) or Proposition (17.6.3)]. Thus, τ is étale along Σ by [EGA, IV, Proposition (17.8.2)], and this completes the proof. \square

In the rest of Section 2, we shall show some local properties of surface singularities that have a linear chain of smooth rational curves as the exceptional divisor for the minimal resolution.

Lemma 2.9. *Let V be the toric surface $\mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$ of type (n, q) defined over \mathbb{k} and let $D_1, D_2, \nu: U \rightarrow V, G_i$ be the same as above, except that the singular point of V is denoted by $\mathbf{0}$ instead of Σ . Then, there are natural isomorphisms:*

$$\begin{aligned} \nu_* \mathcal{O}_U(G_0) &\simeq \mathcal{O}_V(D_2), & \nu_* \mathcal{O}_{G_0}(G_0) &\simeq \mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{O}_{D_2}, \mathcal{O}_V), \\ \nu_* \mathcal{O}_U(G_{l+1}) &\simeq \mathcal{O}_V(D_1), & \nu_* \mathcal{O}_{G_{l+1}}(G_{l+1}) &\simeq \mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{O}_{D_1}, \mathcal{O}_V). \end{aligned}$$

Proof. Let $\mathbb{k}(V) = \mathbb{k}(U)$ be the rational function field of V and U . For a non-zero rational function $\varphi \in \mathbb{k}(V)$, if $\text{div}(\varphi)_V + D_2 \geq 0$ for the principal divisor $\text{div}(\varphi)_V$ on V , then we have

$$n \text{div}(\varphi)_U + nG_0 = \nu^*(\text{div}(\varphi)_V + nD_2) - \sum_{i=1}^l q_i G_i \geq - \sum_{i=1}^l q_i G_i$$

for the principal divisor $\text{div}(\varphi)_U$ on U by applying (II-7) in Remark 2.4.1. Thus, $\text{div}(\varphi)_U + G_0 \geq 0$, since $q_i < n$ for all $1 \leq i \leq l$. On the other hand, if a non-zero rational function $\psi \in \mathbb{k}(U)$ satisfies $\text{div}(\psi)_U + G_0 \geq 0$, then

$$\text{div}(\psi)_V + D_2 = \nu_*(\text{div}(\psi)_U + G_0) \geq 0.$$

Therefore, $H^0(V, \mathcal{O}_V(D_2)) = H^0(U, \mathcal{O}_U(G_0))$, and we have an isomorphism $\nu_* \mathcal{O}_U(G_0) \simeq \mathcal{O}_V(D_2)$. By applying ν_* to the exact sequence $0 \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U(G_0) \rightarrow \mathcal{O}_{G_0}(G_0) \rightarrow 0$, we have another exact sequence

$$0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_V(D_2) \rightarrow \nu_* \mathcal{O}_{G_0}(G_0) \rightarrow 0,$$

since $R^1 \nu_* \mathcal{O}_U = 0$ by the rationality of the toric singularity or by Corollary 2.8.1. Hence, $\nu_* \mathcal{O}_{G_0}(G_0) \simeq \mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{O}_{D_2}, \mathcal{O}_V)$ is derived from $0 \rightarrow \mathcal{O}_V(-D_2) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_{D_2} \rightarrow 0$ by applying $\mathcal{E}xt_{\mathcal{O}_V}^i(\bullet, \mathcal{O}_V)$. The other isomorphisms concerning G_{l+1} and D_1 are obtained by a similar way. \square

Before going to Lemma 2.10, we recall some basics on the “sheaf of logarithmic one-forms” on the toric surface $V = \mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$ and the residue homomorphism. The sheaf $\Omega_{\mathbb{T}_{\mathbf{N}}/\mathbb{k}}^1$ of one-forms on the torus $\mathbb{T}_{\mathbf{N}} = \text{Spec } \mathbb{k}[\mathbf{M}]$ is trivial by the isomorphism

$$\mathbf{M} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{T}_{\mathbf{N}}} \xrightarrow{\sim} \Omega_{\mathbb{T}_{\mathbf{N}}/\mathbb{k}}^1$$

which maps $m \otimes 1$ to $\mathbf{e}(m)^{-1} d\mathbf{e}(m)$ for any $m \in \mathbf{M}$. Regarding $\mathbb{T}_{\mathbf{N}}$ as the open subset $\mathbb{T}_{\mathbf{N}}(\{0\}) = V \setminus D$ of $V = \mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$, where $D = D_1 + D_2$, we can extend the isomorphism to

$$\theta: \mathbf{M} \otimes_{\mathbb{Z}} \mathcal{O}_V \xrightarrow{\sim} \tilde{\Omega}_{V/\mathbb{k}}^1(\log D) := j_* \Omega_{V^\circ/\mathbb{k}}^1(\log D^\circ),$$

where j is the open immersion $V^\circ := V \setminus \{\mathbf{0}\} \subset V$, and D° is the divisor $D|_{V^\circ}$ (cf. [21, (1.12), Proposition], [15, §4.3, Proposition]). Note that V° and D° are non-singular. The logarithmic tangent sheaf

$$\Theta_{V/\mathbb{k}}(-\log D) := \mathcal{H}om_{\mathcal{O}_V}(\tilde{\Omega}_{V/\mathbb{k}}^1(\log D), \mathcal{O}_V)$$

is isomorphic to $\mathbf{N} \otimes_{\mathbb{Z}} \mathcal{O}_V$ by the dual of θ . The double-dual $(\Omega_{V/\mathbb{k}}^1)^{\vee\vee}$ of the sheaf $\Omega_{V/\mathbb{k}}^1$ of relative one-forms is just the kernel of the residue homomorphism

$$\text{Res}: \tilde{\Omega}_{V/\mathbb{k}}^1(\log D) \rightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}.$$

This residue homomorphism is given by the evaluation map

$$\text{ev}: \mathbf{M} \ni m \mapsto (m(e_1), m(e_2)) \in \mathbb{Z} \oplus \mathbb{Z}$$

in the sense that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{M} \otimes_{\mathbb{Z}} \mathcal{O}_V & \xrightarrow{\text{ev} \otimes \text{id}} & (\mathbb{Z} \oplus \mathbb{Z}) \otimes \mathcal{O}_V \\ \theta \downarrow \simeq & & \downarrow \\ \tilde{\Omega}_{V/\mathbb{k}}^1(\log D) & \xrightarrow{\text{Res}} & \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \end{array}$$

where the right vertical arrow is the direct sum of the natural homomorphisms $\mathcal{O}_V \rightarrow \mathcal{O}_{D_1}$ and $\mathcal{O}_V \rightarrow \mathcal{O}_{D_2}$.

Lemma 2.10. *The residue homomorphism Res is surjective if and only if $p \nmid n$, where $p = \text{char } \mathbb{k}$. If $p \mid n$, then the cokernel of Res is the skyscraper sheaf at $\mathbf{0}$ corresponding to the residue field $\mathbb{k}(\mathbf{0})$ of $\mathcal{O}_{V,\mathbf{0}}$, and the image of Res is isomorphic to \mathcal{O}_D .*

Proof. The subgroup \mathbf{M} of $\mathbf{M}_0 = \mathbb{Z}h_1 + \mathbb{Z}h_2$ is generated by elements $a_1h_1 + a_2h_2$ such that $a_1 + qa_2 \equiv 0 \pmod{n}$. Here, $\text{Res} \circ \theta$ maps $(a_1h_1 + a_2h_2) \otimes 1$ to $(a_1, a_2) \in H^0(D_1, \mathcal{O}_{D_1}) \oplus H^0(D_2, \mathcal{O}_{D_2})$. We shall show that the cokernel of Res is isomorphic to $\mathbb{k}(\mathbf{0})/n\mathbb{k}(\mathbf{0})$ by the homomorphism

$$\phi: \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \ni (\alpha_1, \alpha_2) \mapsto (\alpha_1|_{\mathbf{0}} + q\alpha_2|_{\mathbf{0}}) \bmod n \in \mathbb{k}(\mathbf{0})/n\mathbb{k}(\mathbf{0}).$$

The homomorphism ϕ is in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_V(-D_1) \oplus \mathcal{O}_V(-D_2) & \longrightarrow & M_0 \otimes_{\mathbb{Z}} \mathcal{O}_V & \longrightarrow & \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \longrightarrow 0 \\ & & \phi' \downarrow & & \phi'' \downarrow & & \phi \downarrow \\ & & (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{I} & \longrightarrow & \mathcal{O}_V/n\mathcal{O}_V & \longrightarrow & \mathbb{k}(\mathbf{0})/n\mathbb{k}(\mathbf{0}) \longrightarrow 0 \end{array}$$

of exact sequences, where \mathcal{I} is the defining ideal of $\mathbf{0}$, and the homomorphisms ϕ' and ϕ'' are also induced by

$$\mathcal{O}_V \oplus \mathcal{O}_V \ni (\beta_1, \beta_2) \mapsto \beta_1 + q\beta_2 \in \mathcal{O}_V.$$

Here, ϕ' is surjective, since $\mathcal{I} = \mathcal{O}_V(-D_1) + \mathcal{O}(-D_2)$ (cf. Remark 2.2.1) and since $\gcd(n, q) = 1$. The kernel of ϕ'' is just the image of the natural homomorphism $M \otimes_{\mathbb{Z}} \mathcal{O}_V \rightarrow M_0 \otimes \mathcal{O}_V$. Thus, the image of $\text{Res}: M \otimes_{\mathbb{Z}} \mathcal{O}_V \rightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}$ is just the kernel of ϕ . In particular, Res is surjective if and only if $p \nmid n$.

Assume that $p \mid n$. Then, $p \nmid q$ by $\gcd(n, q) = 1$. Hence, the image of Res is also the kernel of

$$\mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \ni (\alpha_1, \alpha_2) \mapsto \alpha_1|_0 + \alpha_2|_0 \in \mathbb{k}(\mathbf{0}).$$

Thus, the image of Res is just \mathcal{O}_D , since we have a natural exact sequence $0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \rightarrow \mathbb{k}(\mathbf{0}) \rightarrow 0$. \square

Proposition 2.11. *Let Y be a normal algebraic surface over an algebraically closed field \mathbb{k} and let $\mu: M \rightarrow Y$ be the minimal resolution of singularities. Assume that any connected component of the μ -exceptional locus E is a linear chain of rational curves. Then, the following hold:*

- (1) *The natural injection $\mu_*\Theta_{M/\mathbb{k}}(-\log E) \rightarrow \mu_*\Theta_{M/\mathbb{k}}$ is an isomorphism.*
- (2) *$R^i \mu_*\Theta_{M/\mathbb{k}}(-\log E) = 0$ for all $i > 0$.*
- (3) *The direct image sheaf $\mu_*\Omega_{M/\mathbb{k}}^1$ is reflexive. In other words, $(\Omega_{Y/\mathbb{k}}^1)^{\vee\vee} \simeq \mu_*\Omega_{M/\mathbb{k}}^1$.*
- (4) *The natural injection $\mu_*\Theta_{M/\mathbb{k}} \hookrightarrow \Theta_{Y/\mathbb{k}}$ is not an isomorphism if and only if $p \mid n$ and $q = n - 1$.*

Proof. Since the assertions are étale local on Y , we may assume that Y is the toric surface $V = \mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$ by Theorem 2.6. Let $D_1, D_2, \mathbf{0}, \nu: U \rightarrow V$, and G_i be as before. Then, the minimal resolution $M \rightarrow Y$ is just $U \rightarrow V$ and $E = G = \sum_{i=1}^l G_i$. We set $\widehat{G} = \sum_{i=0}^{l+1} G_i = G + G_0 + G_{l+1}$. Since G and \widehat{G} are simple normal crossing divisors, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{U/\mathbb{k}}(-\log \widehat{G}) & \longrightarrow & \Theta_{U/\mathbb{k}} & \longrightarrow & \bigoplus_{i=0}^{l+1} \mathcal{O}_{G_i}(G_i) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \Theta_{U/\mathbb{k}}(-\log G) & \longrightarrow & \Theta_{U/\mathbb{k}} & \longrightarrow & \bigoplus_{i=1}^l \mathcal{O}_{G_i}(G_i) \longrightarrow 0 \end{array}$$

of exact sequences. The assertion (1) is derived from the bottom sequence by taking ν_* , since $\nu_*\mathcal{O}_{G_i}(G_i) = H^0(\mathbb{P}^1, \mathcal{O}(-b_i)) = 0$ for $1 \leq i \leq l$. The commutative diagram above induces an exact sequence

$$(II-10) \quad 0 \rightarrow \Theta_{U/\mathbb{k}}(-\log \widehat{G}) \rightarrow \Theta_{U/\mathbb{k}}(-\log G) \rightarrow \mathcal{O}_{G_0}(G_0) \oplus \mathcal{O}_{G_{l+1}}(G_{l+1}) \rightarrow 0.$$

Here, $\Theta_{U/\mathbb{k}}(-\log \widehat{G}) \simeq \mathbf{N} \otimes_{\mathbb{Z}} \mathcal{O}_U$, since \widehat{G} is the complement of the torus $\mathbb{T}_{\mathbf{N}}$ in $U = \mathbb{T}_{\mathbf{N}}(\Delta)$. Now, $R^i \nu_*\mathcal{O}_U = 0$ for $i > 0$, since V has only rational singularities (cf. Corollary 2.8.1).

We have also $R^i \nu_* \mathcal{O}_{G_0}(G_0) = R^i \nu_* \mathcal{O}_{G_{l+1}}(G_{l+1}) = 0$ for $i > 0$, since ν induces isomorphisms $G_0 \rightarrow D_2$ and $G_{l+1} \rightarrow D_1$. Thus, the assertion (2) is obtained by applying $R^i \nu_*$ to the exact sequence (II-10).

For the assertion (3), we consider the commutative diagram

$$\begin{array}{ccc} \nu_* \Omega_{U/\mathbb{k}}^1(\log \widehat{G}) & \xrightarrow{\nu_* \text{Res}} & \bigoplus_{i=0}^{l+1} \nu_* \mathcal{O}_{G_i} \\ \alpha \uparrow & & \uparrow \beta \\ \widetilde{\Omega}_{V/\mathbb{k}}^1(\log D) & \xrightarrow{\text{Res}} & \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \end{array}$$

obtained by comparing the residue homomorphisms on V and U . Here, α is an isomorphism, since $\widetilde{\Omega}_V^1(\log D) \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathcal{O}_V$ and $\Omega_U^1(\log \widehat{G}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathcal{O}_U$. The map β is an isomorphism to $\nu_* \mathcal{O}_{E_{l+1}} \oplus \nu_* \mathcal{O}_{E_0}$. Thus, we have a homomorphism

$$\gamma: (\Omega_{V/\mathbb{k}}^1)^{\vee\vee} \rightarrow \nu_* \Omega_{U/\mathbb{k}}^1$$

as the induced homomorphism between the kernels of the top and bottom homomorphisms. Then, γ is an isomorphism, since it is an isomorphism on $V \setminus \{\mathbf{0}\}$, the source is reflexive, and the target is torsion free. Hence, we have (3).

It remains to prove (4). Let \mathcal{F} be the image of $\text{Res}: \widetilde{\Omega}_{V/\mathbb{k}}^1(\log D) \rightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}$. Then, by (II-10) and by Lemma 2.9, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{V/\mathbb{k}}(-\log D) & \longrightarrow & \nu_* \Theta_{U/\mathbb{k}}(-\log G) & \longrightarrow & \mathcal{E}xt_{\mathcal{O}_V}^1(\bigoplus_{i=1}^2 \mathcal{O}_{D_i}, \mathcal{O}_V) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Theta_{V/\mathbb{k}}(-\log D) & \longrightarrow & \Theta_{V/\mathbb{k}} & \longrightarrow & \mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{F}, \mathcal{O}_V) \longrightarrow 0 \end{array}$$

of exact sequences, in which the bottom one is obtained from

$$0 \rightarrow (\Omega_{V/\mathbb{k}}^1)^{\vee\vee} \rightarrow \widetilde{\Omega}_{V/\mathbb{k}}^1(\log D) \rightarrow \mathcal{F} \rightarrow 0$$

by taking $\mathcal{E}xt_{\mathcal{O}_V}^i(\bullet, \mathcal{O}_V)$. Hence, if $p \nmid n$, then $\mathcal{F} = \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}$ by Lemma 2.10 and $\Theta_{V/\mathbb{k}} \simeq \nu_* \Theta_{U/\mathbb{k}}(-\log G)$ by the commutative diagram above. Thus, we may assume that $p \mid n$. Then, by Lemma 2.10, $\mathcal{F} = \mathcal{O}_D$, and we obtain an exact sequence

$$\begin{aligned} \cdots \rightarrow \mathcal{E}xt_{\mathcal{O}_V}^i(\mathbb{k}(\mathbf{0}), \mathcal{O}_V) &\rightarrow \mathcal{E}xt_{\mathcal{O}_V}^i(\mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}, \mathcal{O}_V) \\ &\rightarrow \mathcal{E}xt_{\mathcal{O}_V}^i(\mathcal{O}_D, \mathcal{O}_V) \rightarrow \mathcal{E}xt_{\mathcal{O}_V}^{i+1}(\mathbb{k}(\mathbf{0}), \mathcal{O}_V) \rightarrow \cdots \end{aligned}$$

It is enough to determine when the cokernel of

$$\mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}, \mathcal{O}_V) \rightarrow \mathcal{E}xt_{\mathcal{O}_V}^1(\mathcal{O}_D, \mathcal{O}_V)$$

is not zero. By a standard argument, we see that the cokernel is isomorphic to that of the homomorphism $\mathcal{O}_V(D_1) \oplus \mathcal{O}_V(D_2) \rightarrow \mathcal{O}_V(D)$ induced from the natural inclusions $\mathcal{O}_V(D_1) \hookrightarrow \mathcal{O}_V(D)$ and $\mathcal{O}_V(D_2) \hookrightarrow \mathcal{O}_V(D)$. Now, $H^0(V, \mathcal{O}_V(D))$ is generated by the

rational functions φ on V such that $\operatorname{div}(\varphi) + D \geq 0$. Hence, $H^0(V, \mathcal{O}_V(D))$ is generated by the following monomials $\mathbf{x}_1^{a_1} \mathbf{x}_2^{a_2}$ as an $H^0(V, \mathcal{O}_V)$ -submodule of $\mathbb{k}[\mathbf{M}] = H^0(\mathbb{T}_N, \mathcal{O})$:

- a_1 and a_2 are integers with $a_1 + qa_2 \equiv 0 \pmod{n}$.
- $a_1 \geq -1$ and $a_2 \geq -1$ (cf. (II-2)).

Similarly, $H^0(V, \mathcal{O}_V(D_1)) + H^0(V, \mathcal{O}_V(D_2))$ is generated by the following monomials $\mathbf{x}_1^{b_1} \mathbf{x}_2^{b_2}$ as an $H^0(V, \mathcal{O}_V)$ -submodule of $\mathbb{k}[\mathbf{M}]$:

- b_1 and b_2 are integers with $b_1 + qb_2 \equiv 0 \pmod{n}$.
- $b_1 \geq -1$ and $b_2 \geq -1$, but $\max\{b_1, b_2\} \geq 0$.

Therefore, the cokernel is not zero if and only if the monomial $\mathbf{x}_1^{-1} \mathbf{x}_2^{-1}$ is contained in $H^0(V, \mathcal{O}_V(D))$. This is just the case where $q = n - 1$. Thus (4) is proved, and we have finished the proof of Proposition 2.11. \square

Remark 2.11.1. If the injection of the assertion (4) is an isomorphism, then μ is an “equivariant resolution of singularities” in the sense of Hironaka. Hence, for toric singularities of type $(n, q = n - 1)$, equivariant resolutions do exist if and only if $p \nmid n$ (cf. [53, Theorem] and an example of characteristic two in [5, page 345]).

Corollary 2.12. *In the situation of Proposition 2.11, one has an isomorphism*

$$H^2(M, \Theta_{M/\mathbb{k}}(-\log E)) \xrightarrow{\sim} H^2(Y, \Theta_{Y/\mathbb{k}}).$$

Proof. Since $\Theta_{Y/\mathbb{k}}$ is the double-dual of $\mu_* \Theta_{M/\mathbb{k}}(-\log E)$, there is an exact sequence

$$0 \rightarrow \mu_* \Theta_{M/\mathbb{k}}(-\log E) \rightarrow \Theta_{Y/\mathbb{k}} \rightarrow \mathcal{G} \rightarrow 0$$

for a coherent sheaf \mathcal{G} with $\dim \operatorname{Supp} \mathcal{G} \leq 0$, which induces an isomorphism

$$\iota_1: H^2(Y, \mu_* \Theta_{M/\mathbb{k}}(-\log E)) \simeq H^2(Y, \Theta_{Y/\mathbb{k}}).$$

On the other hand, since $R^i \mu_* \Theta_{M/\mathbb{k}}(-\log E) = 0$ for all $i > 0$ by Proposition 2.11.(2), the Leray spectral sequence for μ induces an isomorphism

$$i_2: H^2(X, \mu_* \Theta_{M/\mathbb{k}}(-\log E)) \simeq H^2(M, \Theta_{M/\mathbb{k}}(-\log E)).$$

Therefore, we obtain the claimed isomorphism as $i_1 \circ i_2^{-1}$. \square

3. TORIC SINGULARITY OF CLASS T

In this section, we introduce the notion of toric singularity of class T and study its properties. We first discuss some invariants arising from toric singularities of class T: The results here are already known in papers such as [54], [37], [26], [52], [38], [32], but we shall give a self-contained proof. Tables 1 and 2 obtained here are used in some calculations in Sections 6 and 7. Second, in Theorem 3.8 below, we shall construct a special smoothing

(deformation) of toric singularities of class T, which plays an important role in producing new surfaces.

Definition 3.1. Let X be a normal algebraic surface defined over \mathbb{k} and x a closed point. The germ (X, x) (in the étale topology) is said to be a “toric singularity of type (n, q) ” if the formal completion of $\mathcal{O}_{X, x}$ is isomorphic to the formal completion of $\mathcal{O}_{V, \mathbf{0}}$ for an affine toric surface V of type (n, q) (cf. Definition 2.2) over \mathbb{k} and the zero-dimensional orbit $\mathbf{0}$.

Remark 3.1.1. Note that by [2, Corollary 2.6], the condition in Definition 3.1 is equivalent to the existence of a common étale neighborhood of (X, x) and $(V, \mathbf{0})$.

Remark 3.1.2. By Theorem 2.7, (X, x) is a toric singularity if and only if the exceptional locus of the minimal resolution is a linear chain of smooth rational curves.

Definition 3.2. Let \mathcal{T}_{DNA} be the set of triples (d, n, a) of positive integers with $n > a$ and $\gcd(n, a) = 1$. A two-dimensional surface singularity is said to be of type $T(d, n, a)$ for a triplet $(d, n, a) \in \mathcal{T}_{\text{DNA}}$ if it is a toric singularity of type $(dn^2, dna - 1)$. The singularities of “class T” are the singularities of type $T(d, n, a)$ for all $(d, n, a) \in \mathcal{T}_{\text{DNA}}$ (cf. [26, Proposition 3.10], [38, §4]).

Remark 3.2.1. The definition of class T in [26] is different from ours. Our definition of class T corresponds to that of non-Gorenstein class T in [26].

Before going to the study of toric singularities of class T, we prepare some invariants arising from each element of \mathcal{T}_{DNA} . Let (d, n, a) be a triplet in \mathcal{T}_{DNA} . We can define positive integers l, b_1, \dots, b_l by the property that $l \geq 1, b_i \geq 2$ for all $1 \leq i \leq l$, and $dn^2/(dna - 1) = [b_1, \dots, b_l]$. Then, $(b_1, \dots, b_l) \neq (2, 2, \dots, 2)$; for otherwise, $dn^2 = dna$ contradicting $n > a$. By Lemma 2.3, we can define also non-negative integers p_i and q_i for $0 \leq i \leq l + 1$ by the following properties:

- $p_0 = 0 < p_1 = 1 < p_2 < \dots < p_l < p_{l+1} = dn^2$.
- $q_0 = dn^2 > q_1 = dna - 1 > q_2 > \dots > q_l = 1 > q_{l+1} = 0$
- $p_{i-1} + p_{i+1} = b_i p_i$ and $q_{i-1} + q_{i+1} = b_i q_i$ for all $1 \leq i \leq l$.

We set $r_i := (p_i + q_i)/(dn)$ for $0 \leq i \leq l + 1$. Then, $r_0 = r_{l+1} = n$ and $r_1 = a$. Moreover, we have:

Lemma 3.3. (1) r_i is a positive integer with $1 \leq r_i < n$ for all $1 \leq i \leq l$.
 (2) $r_i \equiv ap_i \equiv -aq_i \pmod{n}$ for all $1 \leq i \leq l$.
 (3) $r_l = n - a$, equivalently, $p_l = dn(n - a) - 1$.

Proof. The assertions (1) and (2) follow from the convexity $r_{i-1} + r_{i+1} = b_i r_i$ and $p_i q_1 \equiv q_i \pmod{dn^2}$ for $1 \leq i \leq l$ (cf. Lemma 2.3.(5)). The last assertion (3) is a consequence of the previous two assertions. \square

Definition 3.4. For $(d, n, a) \in \mathcal{T}_{\text{DNA}}$, we define:

$$\begin{aligned} B(d, n, a) &:= (b_1, b_2, \dots, b_l), \quad P(d, n, a) := (p_1, p_2, \dots, p_l), \quad Q(d, n, a) := (q_1, q_2, \dots, q_l), \\ R(d, n, a) &:= (r_1, r_2, \dots, r_l), \quad C(d, n, a) := (c_1, c_2, \dots, c_l) \quad \text{where} \quad c_i := 1 - r_i/n, \\ \delta(d, n, a) &:= \sum_{i=1}^l b_i - (2l + 1), \quad \text{and} \quad l(d, n, a) := l. \end{aligned}$$

The following characterization of toric singularities of class T is well-known:

Lemma 3.5 ([37, Proposition 5.9]). *Let (X, x) be a normal surface singularity such that the exceptional locus of the minimal resolution $\mu: M \rightarrow (X, x)$ of singularity is a linear chain of smooth rational curves. Then, the following two conditions are mutually equivalent:*

- (1) Δ^2 is a negative integer for the effective \mathbb{Q} -divisor $\Delta = \mu^*(K_X) - K_M$.
- (2) (X, x) is a toric surface singularity of class T.

Moreover, if (X, x) is a singularity of type $T(d, n, a)$, then $\Delta^2 = -\delta(d, n, a)$, and $\Delta = \sum_{i=1}^l c_i E_i$ for the linear chain $E_1 + \dots + E_l$ of smooth rational curves and for $C(d, n, a) = (c_1, \dots, c_l)$.

Proof. By Theorem 2.7, we may assume that X is a toric surface V of type (n, q) for some positive integers n, q with $n > q$ and $\gcd(n, q) = 1$. Thus, we can use the description of the minimal resolution $\nu: U \rightarrow V$ of the toric surface given in Section 2. Note that $\nu^*(K_V + D_1 + D_2) = K_U + \sum_{i=0}^{l+1} G_i \sim 0$. Thus, we have

$$\Delta = \nu^*(K_V) - K_U = \sum_{i=0}^{l+1} G_i - \nu^*(D_1 + D_2) = \sum_{i=1}^l \left(1 - \frac{p_i + q_i}{n}\right) G_i$$

for integers p_i, q_i in Lemma 2.3, by (II-6) in Remark 2.4.1. In particular, if $(n, q) = (dm^2, dma - 1)$ for a triplet $(d, m, a) \in \mathcal{T}_{\text{DNA}}$, then we have $\Delta = \sum_{i=1}^l c_i G_i$ for $C(d, m, a) = (c_1, \dots, c_m)$. On the other hand,

$$\Delta G_i = -K_U G_i = 2 + G_i^2 = 2 - b_i$$

for all $1 \leq i \leq l$ by adjunction, where $n/q = [b_1, \dots, b_l]$. Hence,

$$\Delta^2 = \sum_{i=1}^l \left(1 - \frac{p_i + q_i}{n}\right) (2 - b_i),$$

and it is an integer if and only if

$$\sum_{i=1}^l (p_i + q_i)(2 - b_i) \equiv 0 \pmod{n}.$$

Since $b_i(p_i, q_i) = (p_{i-1}, q_{i-1}) + (p_{i+1}, q_{i+1})$ (cf. Lemma 2.3), we have

$$\begin{aligned} \sum_{i=1}^l (p_i + q_i)(2 - b_i) &= 2 \sum_{i=1}^l (p_i + q_i) - \sum_{i=1}^l (p_{i-1} + q_{i-1}) - \sum_{i=1}^l (p_{i+1} + q_{i+1}) \\ &= p_1 + p_l + q_1 + q_l - (p_0 + p_{l+1} + q_0 + q_{l+1}) = q + q' + 2 - 2n, \end{aligned}$$

where $0 < q' < n$ with $qq' \equiv 1 \pmod n$ (cf. Lemma 2.3.(5)). Thus, $\Delta^2 \in \mathbb{Z}$ if and only if $q + q' + 2 \equiv 0 \pmod n$. Since $\gcd(n, q) = 1$, this condition is equivalent to

$$(q + 1)^2 = q^2 + 2q + 1 \equiv q(q + 2 + q') \equiv 0 \pmod n.$$

By considering the prime factorization, we see that this is also equivalent to either

- (i) $n = q + 1$, or
- (ii) $n = dm^2$ and $q + 1 = dma$ for some $(d, m, a) \in \mathcal{T}_{\text{DNA}}$.

In case (i), $(b_1, \dots, b_l) = (2, 2, \dots, 2)$ and $\Delta^2 = 0$. In case (ii), we have $q + q' + 2 = dm^2$ by Lemma 3.3.(3), and

$$\Delta^2 = \sum_{i=1}^l (2 - b_i) - \frac{1}{dm^2}(q + q' + 2 - 2dm^2) = 2l + 1 - \sum_{i=1}^l b_i = -\delta(d, m, a) < 0.$$

Thus, we are done. \square

Corollary 3.5.1. *Let X be a normal projective surface whose non-Gorenstein singularities are toric singularities of class T . Then, K_X^2 is an integer.*

Lemma 3.6 (cf. [26, Proposition 3.11]). *For a triplet $(d, n, a) \in \mathcal{T}_{\text{DNA}}$, either $b_1 \geq 3$ or $b_l \geq 3$ holds for $B(d, n, a) = (b_1, \dots, b_l)$. Assume that $b_1 \geq 3$ and $b_l \geq 3$. Then, $(d, n, a) = (l, 2, 1)$. Here, $B(1, 2, 1) = (4)$, $B(2, 2, 1) = (3, 3)$, and $B(l, 2, 1) = (3, 2, \dots, 2, 3)$ for $l \geq 3$. Moreover, for all $1 \leq i \leq l$,*

$$(p_i, q_i, r_i, c_i) = (2i - 1, 2(l - i) + 1, 1, 1/2).$$

Proof. Assume first that $l = 1$. Then $dn^2 = b_1(dna - 1)$ and $\gcd(dn^2, dna - 1) = 1$ imply that $dn^2 = b_1$ and $dna = 2$. Hence $(d, n, a) = (1, 2, 1)$ and $b_1 = 4$. In this case, $(p_1, q_1, r_1, c_1) = (1, 1, 1, 1/2)$.

Thus, we may assume that $l \geq 2$. Since $p_l = dn(n - a) - 1$ by Lemma 3.3, we have

$$(III-1) \quad \frac{n}{b_1} + \frac{1}{dn} \leq a < \frac{n}{b_1 - 1} + \frac{1}{dn} \quad \text{and} \quad \frac{n}{b_l} + \frac{1}{dn} \leq n - a < \frac{n}{b_l - 1} + \frac{1}{dn}$$

from $0 \leq q_2 = b_1 q_1 - q_0 < q_1$ and $0 \leq p_{l-1} = b_l p_l - p_{l+1} < p_l$. In particular,

$$\frac{1}{b_1} + \frac{1}{b_l} + \frac{2}{dn^2} \leq 1,$$

and hence, $b_1 \geq 3$ or $b_l \geq 3$ holds. Assume that $b_1 \geq 3$ and $b_l \geq 3$. Then,

$$\frac{n}{2} - \frac{1}{dn} \leq n(1 - \frac{1}{b_l - 1}) - \frac{1}{dn} < a < \frac{n}{b_1 - 1} + \frac{1}{dn} \leq \frac{n}{2} + \frac{1}{dn}$$

by (III-1). Since $n \geq 2$ and $a \in \mathbb{Z}$, we have $a = n/2$. Thus, $(n, a) = (2, 1)$, since $\gcd(n, a) = 1$. Therefore, $q_0 = p_{l+1} = 4d$, $q_1 = p_l = 2d - 1$, and hence, $r_0 = r_{l+1} = 2$. For $1 \leq i \leq l$, we have $r_i = 1$, since $0 < r_i < n = 2$ by Lemma 3.3.(1). Thus, $c_i = 1/2$ and $p_i + q_i = 2d$ for all $1 \leq i \leq l$. As a consequence,

$$b_i = \frac{r_{i-1} + r_{i+1}}{r_i} = \begin{cases} 2, & \text{if } 1 < i < l; \\ 3, & \text{if } i = 1 \text{ or } i = l. \end{cases}$$

The equalities $(p_i, q_i) = (2i-1, 2(d-i)+1)$ are shown by induction using $p_{i-1} + p_{i+1} = b_i p_i$ and $q_{i-1} + q_{i+1} = b_i q_i$ with the initial values $p_1 = q_l = 1$ and $p_0 = q_{l+1} = 0$. \square

If $(d, n, a) \in \mathcal{T}_{\text{DNA}}$, then $(d, n, n-a)$, $(d, 2n-a, n)$, $(d, n+a, a) \in \mathcal{T}_{\text{DNA}}$. Thus, we have three maps $\mathbf{i}, \mathbf{t}_L, \mathbf{t}_R: \mathcal{T}_{\text{DNA}} \rightarrow \mathcal{T}_{\text{DNA}}$ defined by

$$\mathbf{i}(d, n, a) = (d, n, n-a), \quad \mathbf{t}_L(d, n, a) = (d, 2n-a, n), \quad \mathbf{t}_R(d, n, a) = (d, n+a, a).$$

Thus, \mathbf{i} is an involution, and $\mathbf{t}_L = \mathbf{i} \circ \mathbf{t}_R \circ \mathbf{i}$. Concerning these maps, we have:

Lemma 3.7 (cf. [54, (2.8.2)], [26, Proposition 3.11], [52, Lemma 3.4], [38, Theorem 17], [32, Theorem 15]). *Let (d, n, a) be a triplet in \mathcal{T}_{DNA} with $B(d, n, a) = (b_1, b_2, \dots, b_l)$ and $R(d, n, a) = (r_1, r_2, \dots, r_l)$. Then, the following hold:*

$$\begin{aligned} B(d, 2n-a, n) &= (2, b_1, \dots, b_{l-1}, b_l + 1), & \delta(d, 2n-a, n) &= \delta(d, n, a) + 1, \\ R(d, 2n-a, n) &= (n = r_1 + r_l, r_1, \dots, r_l), & l(d, 2n-a, n) &= l(d, n, a) + 1, \\ B(d, n+a, a) &= (b_1 + 1, b_2, \dots, b_l, 2), & \delta(d, n+a, a) &= \delta(d, n, a) + 1, \\ R(d, n+a, a) &= (r_1, \dots, r_l, n = r_1 + r_l), & l(d, n+a, a) &= l(d, n, a) + 1. \end{aligned}$$

Proof. We set

$$P(d, n, a) = (p_1, \dots, p_l), \quad Q(d, n, a) = (q_1, \dots, q_l), \quad \text{and} \quad C(d, n, a) = (c_1, \dots, c_l).$$

Concerning the map $\mathbf{i}: (d, n, a) \mapsto (d, n, n-a)$, we have

$$\begin{aligned} B(d, n, n-a) &= (b_l, b_{l-1}, \dots, b_1), & \delta(d, n, n-a) &= \delta(d, n, a), \\ P(d, n, n-a) &= (p_l, p_{l-1}, \dots, p_1), & Q(d, n, n-a) &= (q_l, q_{l-1}, \dots, q_1), \\ R(d, n, n-a) &= (r_l, r_{l-1}, \dots, r_1), & C(d, n, n-a) &= (c_l, c_{l-1}, \dots, c_1), \end{aligned}$$

by $p_l = dn(n-a) - 1$ (cf. Lemma 3.3.(3)). Hence, the equalities for $\mathbf{t}_L(d, n, a) = (d, 2n-a, n)$ are derived from those for $\mathbf{t}_R(d, n, a) = (d, n+a, a)$ by $\mathbf{t}_L = \mathbf{i} \circ \mathbf{t}_R \circ \mathbf{i}$. Thus, it is enough to prove the equalities for $(d, n+a, a)$. We set

$$(p'_0, q'_0) := (0, d(n+a)^2) \quad \text{and} \quad (p'_{l+2}, q'_{l+2}) := (d(n+a)^2, 0),$$

(d, n, a)	δ	l	$\begin{pmatrix} b_1 & b_2 & \cdots & b_l \\ r_1 & r_2 & \cdots & r_l \end{pmatrix}$
$(1, 2, 1)$	1	1	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
$(k, 2, 1)$	1	k	$\begin{pmatrix} 3 & 2 & \cdots & 2 & 3 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$
$(1, m, 1)$	$m - 1$	$m - 1$	$\begin{pmatrix} m+2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & m-1 \end{pmatrix}$
$(1, 2m - 1, m)$	m	m	$\begin{pmatrix} 2 & m+2 & 2 & \cdots & 2 & 3 \\ m & 1 & 2 & \cdots & m-2 & m-1 \end{pmatrix}$
$(k, m, 1)$	$m - 1$	$m + k - 1$	$\begin{pmatrix} m+1 & 2 & \cdots & 2 & 3 & 2 & 2 & \cdots & 2 \\ 1 & 1 & \cdots & 1 & 1 & 2 & 3 & \cdots & m-1 \end{pmatrix}$
$(1, 3m - 1, m)$	$m + 1$	$m + 1$	$\begin{pmatrix} 3 & m+2 & 2 & \cdots & 2 & 3 & 2 \\ m & 1 & 2 & \cdots & m-2 & m-1 & 2m-1 \end{pmatrix}$

(Here, $k \geq 2$ and $m \geq 3$)

TABLE 1. Invariants related to singularities of class T, Part I

and for $1 \leq i \leq l + 1$, we set

$$(III-2) \quad (p'_i, q'_i) := \frac{1}{dn^2}(p_i, q_i) \begin{pmatrix} dn(n+a) - 1 & 1 \\ -1 & dn(n+a) + 1 \end{pmatrix}.$$

Then, p'_i and q'_i are positive integers by Lemma 3.3.(2), and we have

$$(p'_{i-1}, q'_{i-1}) + (p'_{i+1}, q'_{i+1}) = \begin{cases} (b_1 + 1)(p'_1, q'_1), & \text{for } i = 1, \\ b_i(p'_i, q'_i), & \text{for } 2 \leq i \leq l, \\ 2(p'_{l+1}, q'_{l+1}), & \text{for } i = l + 1, \end{cases}$$

for all $1 \leq i \leq l + 1$. Moreover, $q'_1 = da(n+a) - 1$ by (III-2). Therefore, $B(d, n+a, a) = (b_1+1, b_2, \dots, b_l, 2)$, $P(d, n+a, a) = (p'_1, p'_2, \dots, p'_{l+1})$, and $Q(d, n+a, a) = (q'_1, q'_2, \dots, q'_{l+1})$. In particular,

$$\delta(d, n+a, a) = 3 + \sum_{i=1}^l b_i - (2(l+1) + 1) = \delta(d, n+a, a) + 1.$$

By (III-2), we have

$$\frac{p'_i + q'_i}{d(n+a)} = \frac{dn(n+a)(p_i + q_i)}{d(n+a) \cdot dn^2} = \frac{p_i + q_i}{dn} = r_i$$

for all $1 \leq i \leq l + 1$. Hence, $R(d, n+a, a) = (r_1, \dots, r_l, n = r_1 + r_l)$. Thus, we are done. \square

(d, n, a)	δ	l	$\begin{pmatrix} b_1 & b_2 & \cdots & b_l \\ r_1 & r_2 & \cdots & r_l \end{pmatrix}$
$(1, 11, 3)$	5	5	$\begin{pmatrix} 4 & 5 & 3 & 2 & 2 \\ 3 & 1 & 2 & 5 & 8 \end{pmatrix}$
$(1, 19, 5)$	7	7	$\begin{pmatrix} 4 & 7 & 2 & 2 & 3 & 2 & 2 \\ 5 & 1 & 2 & 3 & 4 & 9 & 14 \end{pmatrix}$
$(1, 19, 13)$	8	8	$\begin{pmatrix} 2 & 2 & 9 & 2 & 2 & 2 & 2 & 4 \\ 13 & 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$
$(3, 23, 4)$	8	10	$\begin{pmatrix} 6 & 5 & 2 & 3 & 2 & 3 & 2 & 2 & 2 & 2 \\ 4 & 1 & 1 & 1 & 2 & 3 & 7 & 11 & 15 & 19 \end{pmatrix}$
$(1, 25, 17)$	10	10	$\begin{pmatrix} 2 & 2 & 11 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\ 17 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$
$(1, 35, 6)$	10	10	$\begin{pmatrix} 6 & 8 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 \\ 6 & 1 & 2 & 3 & 4 & 5 & 11 & 17 & 23 & 29 \end{pmatrix}$
$(1, 63, 34)$	11	11	$\begin{pmatrix} 2 & 7 & 7 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 3 \\ 34 & 5 & 1 & 2 & 3 & 4 & 9 & 14 & 19 & 24 & 29 \end{pmatrix}$
$(1, 252, 145)$	13	13	$\begin{pmatrix} 2 & 4 & 6 & 2 & 6 & 2 & 4 & 2 & 2 & 2 & 3 & 2 & 3 \\ 145 & 38 & 7 & 4 & 1 & 2 & 3 & 10 & 17 & 24 & 31 & 69 & 107 \end{pmatrix}$

TABLE 2. Invariants related to singularities of class T, Part II

Corollary 3.7.1 (cf. [26, Proposition 3.11], [38, Theorem 17], [32, Proposition 20]). *Any element (d, n, a) of \mathcal{T}_{DNA} is obtained from $(d, 2, 1)$ by a successive compositions of maps \mathbf{t}_L and \mathbf{t}_R . The number of the compositions equals $\delta(d, n, a) - 1 = l(d, n, a) - d$. In particular, $\sum_{i=1}^l b_i = 3l + 2 - d$.*

Proof. If $b_1 \geq 3$ and $b_l \geq 3$, then $(d, n, a) = (d, 2, 1)$, $l(d, 2, 1) = d$, and $\delta(d, 2, 1) = 1$ by Lemma 3.6. If $b_1 = 2$, then $b_l \geq 3$ by Lemma 3.6, and $(d, n, a) = \mathbf{t}_L(d, a, n - 2a)$ with $l(d, n, a) = l(d, a, n - 2a) + 1$ and $\delta(d, n, a) = \delta(d, a, n - 2a) + 1$ by Lemma 3.7. Similarly, if $b_l = 2$, then $b_1 \geq 3$, and $(d, n, a) = \mathbf{t}_R(d, n - a, a)$ with $l(d, n, a) = l(d, n - a, a) + 1$ and $\delta(d, n, a) = \delta(d, n - a, a) + 1$. Hence, we are done by induction on $l(d, n, a)$. \square

In Tables 1 and 2, we list $\delta = \delta(d, n, a)$, $l = l(d, n, a)$, $B(d, n, a)$, and $R(d, n, a)$ for typical elements $(d, n, a) \in \mathcal{T}_{\text{DNA}}$, some of which are used later. For the numbers c_i , we have:

Corollary 3.7.2 (cf. [32, Corollary 17]). *Let (d, n, a) be a triplet in \mathcal{T}_{DNA} with $B(d, n, a) = (b_1, \dots, b_l)$, $R(d, n, a) = (r_1, \dots, r_l)$, and $C(d, n, a) = (c_1, \dots, c_l)$. If $c_i \leq 1/2$ (or equivalently, $r_i/n \geq 1/2$) for some i and if $n > 2$, then either $b_j = 2$ for all $j \leq i$ or $b_k = 2$ for all $k \leq i$.*

Proof. Assume the contrary. Then, $b_j \geq 3$ and $b_k \geq 3$ for some $j \leq i \leq k$. By Lemma 3.6, we have $b_1 = 2$ or $b_l = 2$. Thus, we may assume that $b_1 \geq 3$ and $b_l = 2$. In particular, we may put $j = 1$. Then, $(d, n, a) = \mathbf{t}_R(d, n - a, a)$, and

$$B(d, n - a, a) = (b_1 - 1, b_2, \dots, b_{l-1}) \quad \text{and} \quad R(d, n - a, a) = (r_1, r_2, \dots, r_{l-1})$$

by Lemma 3.7. In particular,

$$c'_i = 1 - r'_i/(n - a) < 1 - r_i/n = c_i \leq 1/2,$$

where $C(d, n - a, a) = (c'_1, \dots, c'_{l-1})$. Hence, in order to derive a contradiction, we may assume that $k = l - 1$.

If $b_1 - 1 \geq 3$, then $n - a = 2$ and $r_i = 1$ by Lemma 3.6; this is a contradiction, since $r_i/n \leq 1/3$. Thus, $b_1 = 3$ and $(d, n - a, a) = \mathbf{t}_L(d, a, 3a - n)$ by Lemma 3.7. In particular, $2a < n < 3a$. Moreover,

$$B(d, a, 3a - n) = (b_2, \dots, b_{l-2}, b_{l-1} - 1) \quad \text{and} \quad R(d, a, 3a - n) = (r_2, \dots, r_{l-1}).$$

Thus, $r_i < a$ by Lemma 3.3 applied to $(d, a, 3a - n)$, and hence, $r_i/n < a/n < 1/2$; this is a contradiction. \square

Finally in Section 3, we shall prove:

Theorem 3.8. *Let Λ be a complete discrete valuation ring or a field. Let V be a toric Λ -scheme of type $(dn^2, dna - 1)$ (cf. Definition 2.2) for some positive integers d, n, a with $n > a$ and $\gcd(n, a) = 1$. Then, there exist a flat family $\mathcal{V} \rightarrow T$ of normal affine surfaces over an open subset T of the affine line \mathbb{A}_Λ^1 and a section $\sigma: \text{Spec } \Lambda \rightarrow T$ satisfying the following conditions:*

- (1) $\mathcal{V} \times_{T, \sigma} \text{Spec } \Lambda \simeq V$.
- (2) $\mathcal{V} \rightarrow T$ is smooth over $T \setminus \sigma(\text{Spec } \Lambda)$.
- (3) \mathcal{V} is normal, and $rK_{\mathcal{V}}$ is Cartier with $\mathcal{O}_{\mathcal{V}}(rK_{\mathcal{V}})|_V \simeq \mathcal{O}_V(rK_V)$ for any integer r divisible by n .

Remark 3.8.1. Our idea of the proof of Theorem 3.8 is taken from the proof of [40, Proposition 4.19], which treats a special case. However, the proof of Proposition 4.19 contains an error when $p \mid n$. The error is corrected by the present proof.

Remark 3.8.2. Roughly speaking, when Λ is a field, Theorem 3.8 asserts that a toric singularity of class T has a “ \mathbb{Q} -Gorenstein smoothing” (cf. [26]). The proof of Theorem 3.8 is essentially the same as in the proof of [26, Proposition 3.10] when Λ is a field of characteristic zero.

We recall the construction of V in Section 2: We have a free abelian group $\mathbf{N}_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2$ with the base (e_1, e_2) and the cone $\boldsymbol{\sigma} = \text{Cone}(e_1, e_2) \subset \mathbf{N}_0 \otimes \mathbb{R}$ such that $V = \mathbb{T}_{\mathbf{N}}(\boldsymbol{\sigma})$ for a free abelian group $\mathbf{N} = \mathbf{N}_0 + \mathbb{Z}v$ defined by the vector

$$v = \frac{1}{dn^2}(e_1 + (dna - 1)e_2).$$

Let \mathbf{M} and \mathbf{M}_0 be the dual abelian groups of \mathbf{N} and \mathbf{N}_0 , respectively. Then, $R_0 = \Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}_0]$ is a polynomial Λ -algebra generated by two-variables $\mathbf{x}_1, \mathbf{x}_2$ which correspond to the dual basis of (e_1, e_2) . In particular, $V_0 = \mathbb{T}_{\mathbf{N}_0}(\boldsymbol{\sigma}) \simeq \mathbb{A}_\Lambda^2$. The affine coordinate ring $R := H^0(V, \mathcal{O}_V) = \Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}]$ is a Λ -subalgebra of $\Lambda[\mathbf{x}_1, \mathbf{x}_2]$ generated by the monomials $\mathbf{x}_1^{k_1} \mathbf{x}_2^{k_2}$ satisfying $k_1 + (dna - 1)k_2 \equiv 0 \pmod{dn^2}$.

In this situation, we define a subgroup \mathbf{N}_1 of \mathbf{N} by

$$\mathbf{N}_1 := \mathbf{N}_0 + \mathbb{Z}nv = \mathbf{N}_0 + \mathbb{Z}\frac{1}{dn}(e_1 + (dn - 1)e_2).$$

Then, we have a toric Λ -scheme $V_1 = \mathbb{T}_{\mathbf{N}_1}(\boldsymbol{\sigma})$, and finite surjective morphisms $\tau_0: V_0 \rightarrow V_1$ and $\tau: V_1 \rightarrow V$ associated with the inclusions $\mathbf{N}_0 \subset \mathbf{N}_1$ and $\mathbf{N}_1 \subset \mathbf{N}$. The affine coordinate ring $R_1 := H^0(V_1, \mathcal{O}_{V_1}) = \Lambda[\boldsymbol{\sigma}^\vee \cap \mathbf{M}_1]$, where \mathbf{M}_1 is the dual abelian group of \mathbf{N}_1 , is a Λ -subalgebra of $\Lambda[\mathbf{x}_1, \mathbf{x}_2]$ generated by three monomials $\mathbf{x}_1^{dn}, \mathbf{x}_2^{dn}, \mathbf{x}_1\mathbf{x}_2$. Thus,

$$R_1 \simeq \Lambda[\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}]/(\mathbf{u}_1\mathbf{u}_2 - \mathbf{z}^{dn})$$

for three variables $\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}$, where the isomorphism above is defined by $\mathbf{u}_1 = \mathbf{x}_1^{dn}, \mathbf{u}_2 = \mathbf{x}_2^{dn}$, and $\mathbf{z} = \mathbf{x}_1\mathbf{x}_2$. Since n is the index of \mathbf{N}/\mathbf{N}_1 , the group subscheme $\boldsymbol{\mu}_n = \text{Ker}(\mathbb{T}_{\mathbf{N}_1} \rightarrow \mathbb{T}_{\mathbf{N}})$ of $\mathbb{T}_{\mathbf{N}_1}$ acts on V_1 and its quotient scheme is just V . Here, the action of $\boldsymbol{\mu}_n$ on $V_1 = \text{Spec } R_1$ is given by

$$(III-3) \quad (\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}) \mapsto (\mathbf{u}_1 \otimes \mathbf{t}, \mathbf{u}_2 \otimes \mathbf{t}^{-1}, \mathbf{z} \otimes \mathbf{t}^a),$$

where $\boldsymbol{\mu}_n$ is regarded as $\text{Spec } \Lambda[\mathbf{t}, \mathbf{t}^{-1}]/(\mathbf{t}^n - 1)$. The action of $\boldsymbol{\mu}_n$ on V_1 is induced from that on $\text{Spec } \Lambda[\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}]$ given by (III-3). Then, the $\boldsymbol{\mu}_n$ -invariant part R^\sim of $\Lambda[\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}]$ is generated by monomials $\mathbf{u}_1^{m_1} \mathbf{u}_2^{m_2} \mathbf{z}^{m_3}$ such that $m_i \geq 0$ for all $1 \leq i \leq 3$ and $m_1 - m_2 + am_3 \equiv 0 \pmod{n}$. We see that $\text{Spec } R^\sim$ is isomorphic to $\mathbb{T}_{\mathbf{N}^\sim}(\boldsymbol{\sigma}^\sim)$ for the affine toric Λ -scheme $\mathbb{T}_{\mathbf{N}^\sim}(\boldsymbol{\sigma}^\sim)$ of relative dimension three defined as follows: Let \mathbf{N}_1^\sim be a free abelian group $\bigoplus_{i=1}^3 \mathbb{Z}e_i^\sim$ of rank three and let $\boldsymbol{\sigma}^\sim$ be the cone $\sum_{i=1}^3 \mathbb{R}_{\geq 0}e_i^\sim$. The abelian group \mathbf{N}^\sim is a subgroup of $\mathbf{N}_1^\sim \otimes \mathbb{Q}$ defined by

$$\mathbf{N}^\sim := \mathbf{N}_1^\sim + \mathbb{Z}\frac{1}{n}(e_1^\sim - e_2^\sim + ae_3^\sim).$$

Then, $V = \operatorname{Spec} R$ is a hypersurface (or a Cartier divisor) of $V^\sim := \operatorname{Spec} R^\sim$ defined by the principal ideal $(u_1 u_2 - z^{dn})$.

Lemma 3.9. *The smallest integer r such that rK_V (resp. rK_{V^\sim}) is Cartier, is n . Thus, n is equal to the Gorenstein index of V (resp. V^\sim).*

Proof. Let (k_1, k_2, k_3) be an integral vector such that $k_1 - k_2 + ak_3 \equiv 0 \pmod{n}$. Then, the principal divisor associated to the monomial $u_1^{k_1} u_2^{k_2} z^{k_3}$ on V is expressed as

$$\operatorname{div}(u_1^{k_1} u_2^{k_2} z^{k_3})_V = (k_1 dn + k_3)D_1 + (k_2 dn + k_3)D_2$$

by (II-6) in Remark 2.4.1. Note that $D := D_1 + D_2 \sim -K_V$. Hence, n is the Gorenstein index of V . In fact, $nD = \operatorname{div}(z^n)$ is Cartier, and if jD is an Cartier divisor for an integer j , then $j = k_1 dn + k_3 = k_2 dn + k_3$ for some integral vector (k_1, k_2, k_3) above; hence $k_1 = k_2$ and $j \equiv k_3 \equiv 0 \pmod{n}$.

For $1 \leq i \leq 3$, let D_i^\sim be the \mathbb{T}_{N^\sim} -invariant divisor on V^\sim corresponding to the ray $\mathbb{R}_{\geq 0} e_i^\sim$. Since $\gcd(n, a) = 1$, we see that

$$\mathbb{R}_{\geq 0} e_i^\sim \cap N^\sim = \mathbb{Z}_{\geq 0} e_i^\sim$$

for $1 \leq i \leq 3$. Then, we have the following equality similar to (II-6):

$$\operatorname{div}(u_1^{k_1} u_2^{k_2} z^{k_3})_{V^\sim} = k_1 D_1^\sim + k_2 D_2^\sim + k_3 D_3^\sim.$$

Here, we know also $D^\sim := \sum_{i=1}^3 D_i^\sim \sim -K_{V^\sim}$ and that nD_i^\sim is Cartier for all i . Suppose that jD^\sim is Cartier for an integer j . Then, $j = k_1 = k_2 = k_3$ for an integral vector (k_1, k_2, k_3) such that $k_1 - k_2 + ak_3 \equiv 0 \pmod{n}$; hence $j \equiv 0 \pmod{n}$. Therefore, the Gorenstein index of V^\sim is also n . \square

Remark 3.9.1. V and V^\sim are \mathbb{Q} -factorial, since the cones σ and σ^\sim are simplicial.

Remark 3.9.2. It is well-known in characteristic zero that the singularity on V^\sim is a cyclic quotient terminal singularity of type $\frac{1}{n}(1, -1, a)$ (cf. [45, §3], [46, (4.13)], [47, (5.1), (5.2)]). Even in the positive characteristic case, the singularity is “terminal” in the sense that there is a resolution $\mu^\sim: M^\sim \rightarrow V^\sim$ of singularity such that

$$K_{M^\sim} = \mu^{\sim*}(K_{V^\sim}) + \sum a_i E_i^\sim$$

for μ^\sim -exceptional prime divisors E_i^\sim and positive rational numbers a_i . In fact, a toric resolution of V^\sim is taken independently of the characteristic, and the discrepancy a_i is also independent of the characteristic.

Remark 3.9.3. If X is a \mathbb{Q} -Gorenstein normal algebraic Λ -scheme and if Y is a normal Cartier divisor of X , then $r(K_X + Y)|_Y \sim rK_Y$ for the Gorenstein index r of X . Indeed,

the left hand side is Cartier and is linearly equivalent to the right hand side on the non-singular locus of Y . In particular, Y is also \mathbb{Q} -Gorenstein and r is divisible by the \mathbb{Q} -Gorenstein index of Y . Applying this to Lemma 3.9, we see that the restriction $nK_{V^\sim}|_V$ is linearly equivalent to nK_V .

Proof of Theorem 3.8. We consider an algebra

$$R_1^\sharp := \Lambda[u_1, u_2, z, s]/(z^{dn} - u_1 u_2 - s(z^n + 1))$$

over the polynomial ring $\Lambda[s]$ of one variable. This is flat over $\Lambda[s]$, since it is Cohen–Macaulay and every fiber of $\text{Spec } R_1^\sharp \rightarrow \text{Spec } \Lambda[s]$ is equidimensional. We consider the μ_n -action on $\text{Spec } R_1^\sharp$ given by

$$(u_1, u_2, z, s) \mapsto (tu_1, t^{-1}u_2, t^a z, s),$$

where $\mu_n = \text{Spec } \Lambda[t, t^{-1}]/(t^n - 1)$. Let R^\sharp be the μ_n -invariant subring, which is the $\Lambda[s]$ -submodule generated by monomials $u_1^{k_1} u_2^{k_2} z^{k_3}$ with $k_1 - k_2 + ak_3 \equiv 0 \pmod n$. We set $V^\sharp := \text{Spec } R^\sharp$, $S := \text{Spec } \Lambda[s] = \mathbb{A}_\Lambda^1$, and let $\sigma: \text{Spec } \Lambda \rightarrow S$ be the section defined by $\Lambda[s] \rightarrow \Lambda[s]/(s) \simeq \Lambda$. Then, $V^\sharp \times_{S, \sigma} \text{Spec } \Lambda \simeq V$. Since R^\sharp is the μ_n -invariant ring of R_1^\sharp , which is a direct summand of R_1^\sharp , we see that V^\sharp is normal and $V^\sharp \rightarrow S$ is flat. On the other hand, the affine scheme V^\sharp is isomorphic to the hypersurface of $V^\sim \times_{\text{Spec } \Lambda} S$ defined by $z^{dn} - u_1 u_2 - s(z^n + 1) = 0$. By Lemma 3.9, $V^\sim \times_{\text{Spec } \Lambda} S$ is \mathbb{Q} -Gorenstein with index n . Hence, V^\sharp is also \mathbb{Q} -Gorenstein with index r dividing n , since V^\sharp is normal (cf. Remark 3.9.3). Here, $r = n$ by Lemma 3.9, since V is also a hypersurface of V^\sharp . Consequently, we have $\mathcal{O}_{V^\sharp}(nK_{V^\sharp})|_V \sim \mathcal{O}_V(nK_V)$ (cf. Remark 3.9.3).

We shall study the singularity of fibers of $V^\sharp \rightarrow S$ outside the section $\sigma(\text{Spec } \Lambda)$ defined by $s = 0$. Let $R^\sharp\langle 1 \rangle$ be the affine coordinate ring of the affine open subset $\{u_1 \neq 0, s \neq 0\}$ of $\text{Spec } R^\sharp$. Then, $R^\sharp\langle 1 \rangle$ is isomorphic to

$$\Lambda[y_1^{\pm 1}, y_2, y_3, s^{\pm 1}]/(y_3^{dn} y_1^{ad} - y_2 - s(y_3^n y_1^a + 1)) \simeq \Lambda[y_1^{\pm 1}, y_3, s^{\pm 1}]$$

by the correspondence $(y_1, y_2, y_3) = (u_1^n, u_1 u_2, zu_1^{-a})$. Thus, $R^\sharp\langle 1 \rangle$ is smooth over $\Lambda[s^{\pm 1}]$. Similarly, the affine coordinate ring $R^\sharp\langle 2 \rangle$ of $\{u_2 \neq 0, s \neq 0\}$ is smooth over $\Lambda[s^{\pm 1}]$, since it is isomorphic to

$$\Lambda[y_1, y_2^{\pm 1}, y_3, s^{\pm 1}]/(y_3^{dn} y_2^{-ad} - y_1 - s(y_3^n y_2^{-a} + 1)) \simeq \Lambda[y_2^{\pm 1}, y_3, s^{\pm 1}]$$

by the correspondence $(y_1, y_2, y_3) = (u_1 u_2, u_2^n, zu_2^a)$.

The affine coordinate ring $R^\sharp\langle 3 \rangle$ of $\{z \neq 0, s \neq 0\}$ is isomorphic to

$$\Lambda[y_1, y_2, y_3^{\pm 1}, s^{\pm 1}]/(y_3^d - y_1 y_2 - s(y_3 + 1))$$

by the correspondence $(y_1, y_2, y_3) = (u_1 z^{-a'}, u_2 z^{a'}, z^n)$, where $0 < a' < n$ with $aa' \equiv 1 \pmod n$. Let p be the characteristic of the residue field of Λ . If $p \mid d$ or $p \mid d-1$, then $R^\sharp\langle 3 \rangle$

is smooth over $\Lambda[\mathbf{s}^{\pm 1}]$ by the Jacobian criterion. If $p \nmid d$ and $p \nmid d-1$, then $R^\sharp\langle 3 \rangle$ is smooth over $\Lambda[\mathbf{s}^{\pm 1}, (\mathbf{s} - c)^{-1}]$ for $c = d^d/(d-1)^{d-1}$ by the Jacobian criterion. Thus, the open subset $T := \operatorname{Spec} \Lambda[\mathbf{s}, (\mathbf{s} - c)^{-1}]$ of $S = \mathbb{A}_\Lambda^1$, $\sigma: \operatorname{Spec} \Lambda \rightarrow T \subset S$, and $\mathcal{V} := V^\sharp \times_S T \rightarrow T$ satisfy all the conditions of Theorem 3.8. \square

4. REVIEW OF DEFORMATION THEORY

We review the deformation theory of \mathbb{k} -schemes for the fixed algebraically closed field \mathbb{k} .

Definition 4.1. Let X be a \mathbb{k} -scheme. Let T be a scheme and let o be a \mathbb{k} -rational point of T , which is just a morphism $o: \operatorname{Spec} \mathbb{k} \rightarrow T$. A *deformation* of X over T with the reference point o is a pair $(Y/T, \iota)$ of a flat morphism $Y \rightarrow T$ of schemes and an isomorphism $\iota: Y \times_{T, o} \operatorname{Spec} \mathbb{k} \simeq X$ over $\operatorname{Spec} \mathbb{k}$.

In this section, we fix a Noetherian local ring Λ which is either \mathbb{k} or a complete discrete valuation ring with residue field \mathbb{k} . For example, the ring of Witt vectors of \mathbb{k} is a candidate of Λ .

We recall some important notions mainly from Schlessinger's article [49]. Let \mathcal{C}_Λ be the category of Artinian local Λ -algebras with residue field \mathbb{k} and let $\widehat{\mathcal{C}}_\Lambda$ be the category of complete Noetherian local Λ -algebras $\mathfrak{R} = (\mathfrak{R}, \mathfrak{m}_\mathfrak{R})$ such that $\mathfrak{R}/\mathfrak{m}_\mathfrak{R}^n \in \mathcal{C}_\Lambda$ for all n . An object \mathfrak{R} of $\widehat{\mathcal{C}}_\Lambda$ defines a functor $h_\mathfrak{R}: \mathcal{C}_\Lambda \rightarrow (\text{Sets})$ to the category of sets by $h_\mathfrak{R}(A) = \operatorname{Hom}_{\widehat{\mathcal{C}}_\Lambda}(\mathfrak{R}, A)$. For two functors $F, G: \mathcal{C}_\Lambda \rightarrow (\text{Sets})$, a morphism $\phi: F \rightarrow G$ means a natural transformation of functors. The morphism $\phi: F \rightarrow G$ is called *smooth* (in the sense of Schlessinger [49, (2.2)]) if the natural map $F(B) \rightarrow F(A) \times_{G(A)} G(B)$ is surjective for any surjection $B \rightarrow A$ in \mathcal{C}_Λ .

The deformation functor $\operatorname{Def}_X: \mathcal{C}_\Lambda \rightarrow (\text{Sets})$ of a \mathbb{k} -scheme X is defined as follows. An infinitesimal deformation of X to an algebra A of \mathcal{C}_Λ is a deformation (X_A, ι) of X over $\operatorname{Spec} A$ with \mathfrak{m}_A as a \mathbb{k} -rational reference point. Here, X_A is a flat A -scheme and ι is an isomorphism $X_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{k} \simeq X$. Another deformation (X'_A, ι') of X is isomorphic to (X_A, ι) if there is a morphism $\phi: X_A \rightarrow X'_A$ over $\operatorname{Spec} A$ such that $\iota' = \phi \circ \iota$. Note that this morphism ϕ is indeed an isomorphism over $\operatorname{Spec} A$, since X_A and X'_A are both homeomorphic to X and since ϕ induces the identity on X . We define $\operatorname{Def}_X(A)$ to be the set of isomorphism classes of deformations of X to A . Then, Def_X gives rise to a functor $\mathcal{C}_\Lambda \rightarrow (\text{Sets})$, which is called the *deformation functor* of X . When X is an affine scheme $\operatorname{Spec} R$, we write Def_R for Def_X .

Definition 4.2 (Pro-couple, formal deformation, and hull). Let \mathfrak{R} be an object of $\widehat{\mathcal{C}}_\Lambda$ and set $\mathfrak{R}_n := \mathfrak{R}/\mathfrak{m}_{\mathfrak{R}}^{n+1}$ for $n \geq 0$. Let ξ be an element of

$$\widehat{\mathrm{Def}}_X(\mathfrak{R}) := \varprojlim_n \mathrm{Def}_X(\mathfrak{R}_n).$$

Note that to give an element of $\widehat{\mathrm{Def}}_X(\mathfrak{R})$ is equivalent to giving a morphism $h_{\mathfrak{R}} \rightarrow \mathrm{Def}_X$ of functors. In [49], the pair (\mathfrak{R}, ξ) is called a *pro-couple*. For each n , the element ξ defines a flat \mathfrak{R}_n -scheme X_n with an isomorphism $\iota_n: X_n \times_{\mathrm{Spec} \mathfrak{R}_n} \mathrm{Spec} \mathbb{k} \simeq X$. Moreover, (X_n, ι_n) form an inductive system. Thus, we have a formal scheme $\mathfrak{X} = \mathfrak{X}_\xi := \varinjlim X_n$ flat over the affine formal scheme $\mathrm{Spf} \mathfrak{R}$ with an isomorphism $\iota: \mathfrak{X} \times_{\mathrm{Spf} \mathfrak{R}} \mathrm{Spec} \mathbb{k} \simeq X$ (cf. [EGA, I, Proposition (10.6.3)]). The pair $(\mathfrak{X}/\mathrm{Spf} \mathfrak{R}, \iota)$ or the system $(X_n, \iota_n)_{n \geq 0}$ is called a *formal deformation* of X over \mathfrak{R} (or over $\mathrm{Spf} \mathfrak{R}$).

Remark 4.2.1. In the definition above, if X is an algebraic \mathbb{k} -scheme, i.e., a \mathbb{k} -scheme of finite type, then \mathfrak{X} is Noetherian and $\mathfrak{X} \rightarrow \mathrm{Spf} \mathfrak{R}$ is a morphism of finite type by [EGA, I, Corollaire (10.6.4), Proposition (10.13.1), Définition (10.13.3)]. If X is proper over \mathbb{k} , then $\mathfrak{X} \rightarrow \mathrm{Spf} \mathfrak{R}$ is proper (cf. [EGA, III, (3.4.1)]).

Definition 4.3. Let (\mathfrak{R}, ξ) be a pro-couple of Def_X and let $\mathfrak{X} \rightarrow \mathrm{Spf} \mathfrak{R}$ be the formal deformation of X associated to (\mathfrak{R}, ξ) .

- (1) The pro-couple (or the formal deformation) is said to be *effective* if there is a flat \mathfrak{R} -scheme \mathcal{X} such that \mathcal{X} is a deformation of X over $\mathrm{Spec} \mathfrak{R}$ and that (\mathfrak{R}, ξ) is induced from the inductive system $\mathcal{X}_n = \mathcal{X} \times_{\mathrm{Spec} \mathfrak{R}} \mathrm{Spec} \mathfrak{R}_n$; equivalently the $\mathfrak{m}_{\mathfrak{R}}$ -adic completion of \mathcal{X} is isomorphic to \mathfrak{X} as a formal scheme over $\mathrm{Spf} \mathfrak{R}$.
- (2) The pro-couple (or the formal deformation) is said to be *algebraizable* if there is a deformation $Y \rightarrow T$ of X with a \mathbb{k} -rational reference point $o \in T$ such that
 - T is a scheme of finite type over Λ ,
 - the completion of the local ring $\mathcal{O}_{T,o}$ is isomorphic to \mathfrak{R} , and
 - the formal completion of Y along the fiber X is isomorphic to \mathfrak{X} over $\mathrm{Spf} \mathfrak{R}$.
- (3) The pro-couple (\mathfrak{R}, ξ) is called a *pro-representable hull* (or a *hull*, for short) of Def_X if the morphism $h_{\mathfrak{R}} \rightarrow \mathrm{Def}_X$ corresponding to ξ is smooth and induces the bijection

$$\mathbf{t}(h_{\mathfrak{R}}) := h_{\mathfrak{R}}(\mathbb{k}[\varepsilon]/(\varepsilon^2)) \rightarrow \mathbf{t}(\mathrm{Def}_X) := \mathrm{Def}_X(\mathbb{k}[\varepsilon]/(\varepsilon^2))$$

between the tangent spaces (cf. [49, Definition 2.7]).

Remark 4.3.1. The hull is unique up to non-canonical isomorphism (cf. [49, Proposition 2.9]). The existence of hull of Def_X is known in the cases when X is proper over \mathbb{k} and when X is affine with only isolated singularities (cf. [49, Proposition 3.10]).

Remark 4.3.2. Let X be a projective \mathbb{k} -scheme and let $\mathfrak{X} \rightarrow \mathrm{Spf} \mathfrak{R}$ be the formal deformation associated with a pro-representable hull (\mathfrak{R}, ξ) of Def_X . If \mathfrak{X} admits a relatively ample invertible sheaf over $\mathrm{Spf} \mathfrak{R}$, then this is effective by a projective morphism $\mathcal{X} \rightarrow \mathrm{Spec} \mathfrak{R}$, by an application [EGA, III, Théorème (5.4.5)] of Grothendieck's existence theorem, i.e., \mathfrak{X} is the $\mathfrak{m}_{\mathfrak{R}}$ -adic completion of \mathcal{X} . Moreover, this is algebraizable by Artin's result [3, Theorem 1.6].

Remark 4.3.3. Let X be an equidimensional affine algebraic \mathbb{k} -scheme with only isolated singularities. Then, the pro-representable hull of Def_X is effective by [14, Chapitre IV, Théorème 7] and is algebraizable by [3, Theorem 1.6].

Lemma 4.4. *If X is an affine algebraic \mathbb{k} -scheme with a unique singular point x and if (X', x') is an étale neighborhood of (X, x) , then there is a smooth morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_{\mathcal{O}_{X', x'}}$ of functors on \mathcal{C}_Λ inducing an isomorphism between the tangent spaces.*

Proof. We have a morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_{\mathcal{O}_{X', x'}}$ by [EGA, IV, Théorème (18.1.2)]. The smoothness and the isomorphism between the tangent spaces are derived from [48, Theorem 4.10.(b)]. \square

Remark 4.4.1. If (X, x) is non-singular, then $\mathrm{Def}_{\mathcal{O}_{X, x}}(A)$ consists of one element for any $A \in \mathcal{C}_\Lambda$. In other words, the canonical morphism $\mathrm{Def}_R \rightarrow h_\Lambda$ of functors is an isomorphism. This follows for example from [EGA, IV, Proposition (18.1.1)] and the formal smoothness (cf. [EGA, IV, Définition (17.1.1)]) of $\mathrm{Spec} \mathcal{O}_{X, x} \rightarrow \mathrm{Spec} \mathbb{k}$.

Lemma 4.4.2. *Let (X, x) and (X', x') be as in Lemma 4.4. Let $Y \rightarrow T$ be a deformation of $\mathrm{Spec} \mathcal{O}_{X, x}$ over a scheme T which is with a \mathbb{k} -rational reference point $o \in T$ such that Y is an affine scheme of a local ring. Then, there exist a deformation $Y' \rightarrow T$ of $\mathrm{Spec} \mathcal{O}_{X', x'}$ over T with reference point o and a formally étale morphism $Y' \rightarrow Y$ over T inducing $\mathrm{Spec} \mathcal{O}_{X', x'} \rightarrow \mathrm{Spec} \mathcal{O}_{X, x}$ as a morphism between the fibers over o .*

Proof. This follows from [EGA, IV, Proposition (18.1.1)], since $\mathcal{O}_{X, x}$ and $\mathcal{O}_{X', x'}$ are essentially of finite type over \mathbb{k} . \square

Definition 4.5. Let X be an algebraic \mathbb{k} -scheme and P a \mathbb{k} -rational point. We write $\mathrm{Def}_{(X, P)} := \mathrm{Def}_{\mathcal{O}_{X, P}}$. When X is non-singular outside a finite set (i.e., has only isolated singularities), we define a functor $\mathrm{Def}_X^{(\mathrm{loc})}$ on \mathcal{C}_Λ by

$$\mathrm{Def}_X^{(\mathrm{loc})}(A) := \prod_{P \in \mathrm{Sing} X} \mathrm{Def}_{(X, P)}(A)$$

for $A \in \mathcal{C}_\Lambda$, where $\mathrm{Sing} X$ stands for the singular locus.

Remark 4.5.1. There is a natural morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_{(X,P)}$ of functors for any \mathbb{k} -rational point $P \in X$ which sends $(X_A, \iota) \in \mathrm{Def}_X(A)$ to the pair consisting of the local A -algebra $\mathcal{O}_{X_A, \iota(P)}$ and the isomorphism $\mathcal{O}_{X,P} \simeq \mathcal{O}_{X_A, \iota(P)} \otimes_A \mathbb{k}$ induced by ι . As a consequence, we have a natural morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_X^{(\mathrm{loc})}$ of functors when X has only isolated singularities.

We shall give a proof of the following well-known:

Theorem 4.6 ([55, Proposition 6.4]). *Let X be an algebraic \mathbb{k} -scheme with only isolated singularities. Assume that $H^2(X, \Theta_{X/\mathbb{k}}) = 0$, where $\Theta_{X/\mathbb{k}}$ denotes the tangent sheaf $\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/\mathbb{k}}^1, \mathcal{O}_X)$. Then, the morphism $\mathrm{Def}_X \rightarrow \mathrm{Def}_X^{(\mathrm{loc})}$ of functors is smooth.*

Proof. Let $B \rightarrow A$ be a surjection in \mathcal{C}_A with the kernel I satisfying $I\mathfrak{m}_B = 0$. It suffices to prove that

$$(IV-1) \quad \mathrm{Def}_X(B) \rightarrow \mathrm{Def}_X(A) \times_{\mathrm{Def}_X^{(\mathrm{loc})}(A)} \mathrm{Def}_X^{(\mathrm{loc})}(B)$$

is surjective. An element of the right-hand side of (IV-1) consists of

- a flat A -scheme X_A with an isomorphism $\iota_A: X_A \times_{\mathrm{Spec} A} \mathrm{Spec} \mathbb{k} \simeq X$, and
- flat B -algebras $S_B^{(P)}$ for any $P \in \mathrm{Sing} X$ with isomorphisms $\iota_B^{(P)}: S_B^{(P)} \otimes_B \mathbb{k} \simeq \mathcal{O}_{X,P}$,

where, for any $P \in \mathrm{Sing} X$, we can find an isomorphism $\Psi_P: \mathcal{O}_{X_A, P} \xrightarrow{\sim} S_B^{(P)} \otimes_B A$ such that the composite

$$\mathcal{O}_{X_A, P} \xrightarrow{\Psi_P} S_B^{(P)} \otimes_B A \rightarrow S_B^{(P)} \otimes_B \mathbb{k} \xrightarrow{\iota_B^{(P)}} \mathcal{O}_{X, P}$$

is the homomorphism induced by ι_A .

We apply the obstruction theory of infinitesimal deformations using the cotangent complexes (cf. [20], [34]): Let $\mathbb{L}_{Z/Y}$ be the cotangent complex (as an object of the derived category $D^-(\mathrm{Coh}(Z))$) for a morphism $Z \rightarrow Y$ of schemes. For a coherent sheaf \mathcal{F} on Z , we denote the associated cohomology groups/sheaves by

$$\begin{aligned} T^i(Z/Y, \mathcal{F}) &= \mathbb{E}xt^i(\mathbb{L}_{Z/Y}, \mathcal{F}) = H^i(\mathrm{RHom}_{\mathcal{O}_Z}(\mathbb{L}_{Z/Y}, \mathcal{F})), \\ \mathcal{T}^i(Z/Y, \mathcal{F}) &= \underline{\mathbb{E}xt}^i(\mathbb{L}_{Z/Y}, \mathcal{F}) = \mathcal{H}^i(\mathcal{R}\mathcal{H}om_{\mathcal{O}_Z}(\mathbb{L}_{Z/Y}, \mathcal{F})) \end{aligned}$$

for $i \geq 0$. When Y is the affine scheme $\mathrm{Spec} A$, we write “/ A ” instead of “/ $\mathrm{Spec} A$ ”. If Z is also affine in addition, then Z is replaced with the coordinate ring. We recall a few properties on $T^i(Z/Y, \mathcal{F})$ and $\mathcal{T}^i(Z/Y, \mathcal{F})$.

- (i) $T^0(Z/Y, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{O}_Z}(\Omega_{Z/Y}^1, \mathcal{F})$ (cf. [20, Chapitre II, (1.2.7.4) and Corollaire 1.2.4.3]).
- (ii) If $Z \rightarrow Y$ is smooth, then $T^i(Z/Y, \mathcal{F}) = \mathcal{T}^i(Z/Y, \mathcal{F}) = 0$ for all $i > 0$ (cf. [20, Chapitre III, Proposition 3.1.2]).

- (iii) If $Z \rightarrow Y$ is the base change of a flat morphism $Z_1 \rightarrow Y_1$ by an affine morphism $Y \rightarrow Y_1$, then $T^i(Z/Y, \mathcal{F}) \simeq T^i(Z_1/Y_1, u_*\mathcal{F})$ for the induced affine morphism $u: Z \rightarrow Z_1$ (cf. [34, 2.3.2], [20, Chapitre II, Corollaire 2.3.11]).

We want to find a B -scheme X_B , whose structure sheaf \mathcal{O}_{X_B} is sitting inside a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I\mathcal{O}_{X_A} & \longrightarrow & \mathcal{O}_{X_B} & \longrightarrow & \mathcal{O}_{X_A} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \end{array}$$

of algebra extensions, where the top exact sequence means that the ideal sheaf of X_A in X_B is square zero and is isomorphic to $I\mathcal{O}_{X_A}$ as an \mathcal{O}_{X_A} -module, and where the vertical arrows are natural homomorphisms for the A -scheme X_A and the B -scheme X_B . Note that $I\mathcal{O}_{X_A} \simeq I \otimes_{\mathbb{k}} \mathcal{O}_X$, since $I\mathfrak{m}_B = 0$. We write $I\mathcal{O}_X := I \otimes_{\mathbb{k}} \mathcal{O}_X$. The obstruction class $\text{ob}(X_A)$ for the existence of X_B lies in the cohomology group $T^2(X_A/A, I\mathcal{O}_{X_A})$ (cf. [20, Chapitre III, Théorème 2.1.7.(i)]) which is isomorphic to $T^2(X/\mathbb{k}, I\mathcal{O}_X)$ by (iii) above. We consider the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{T}^q(X/\mathbb{k}, I\mathcal{O}_X)) \Rightarrow E^{p+q} = T^{p+q}(X/\mathbb{k}, I\mathcal{O}_X).$$

Then, $E_2^{2,0} = 0$ by $H^2(X, \Theta_{X/\mathbb{k}}) = 0$, since $\mathcal{T}^0(X/\mathbb{k}, I\mathcal{O}_X) = \Theta_{X/\mathbb{k}} \otimes_{\mathbb{k}} I$ by (i). Moreover, $E_2^{1,1} = 0$, since $\text{Sing } X$ is finite and since $\mathcal{T}^1(X/\mathbb{k}, I\mathcal{O}_X) = \mathcal{T}^1(X/\mathbb{k}, \mathcal{O}_X) \otimes_{\mathbb{k}} I$ is supported on $\text{Sing } X$ by (ii). In particular,

$$T^1(X/\mathbb{k}, I\mathcal{O}_X) = E^1 \rightarrow E_2^{0,1} = \prod_{P \in \text{Sing } X} T^1(\mathcal{O}_{X,P}/\mathbb{k}, I\mathcal{O}_{X,P})$$

is surjective, and

$$T^2(X/\mathbb{k}, I\mathcal{O}_X) = E^2 \rightarrow E_2^{0,2} = \prod_{P \in \text{Sing } X} T^2(\mathcal{O}_{X,P}/\mathbb{k}, I\mathcal{O}_{X,P})$$

is injective. The class $\text{ob}(X_A)$ lies in the kernel of $E^2 \rightarrow E_2^{0,2}$, since $S_A^{(P)}$ possesses a lifting $S_B^{(P)}$ to B . Thus, $\text{ob}(X_A) = 0$. As a consequence, we have a flat B -scheme X_B with an isomorphism $\iota_{B/A}: X_B \times_{\text{Spec } B} \text{Spec } A \simeq X_A$ over A . In other words, we have an element (X_B, ι_B) of $\text{Def}_X(B)$ which is mapped to $(X_A, \iota_A) \in \text{Def}_X(A)$. However, $\mathcal{O}_{X_B,P}$ may not be isomorphic to $S_B^{(P)}$ as a lift of $S_A^{(P)}$ to B . But, by a usual obstruction theory of deformations, the difference of two lifts lies in $T^1(S_A^{(P)}/A, I\mathcal{O}_{X,P}) \simeq T^1(\mathcal{O}_{X,P}/\mathbb{k}, I\mathcal{O}_{X,P})$ (cf. [20, Chapitre III, Théorème 2.1.7.(ii)]). Since $E^1 \rightarrow E_2^{0,1}$ is surjective, we can replace the lift X_B by another one so that $\mathcal{O}_{X_B,P}$ is isomorphic to $S_B^{(P)}$ for any $P \in \text{Sing } X$. Thus, (X_B, ι_B) is mapped to the given element of the right-hand side of (IV-1). Therefore, (IV-1) is surjective. \square

Finally in Section 4, we shall give the following result on algebraization related to Theorem 4.6. Roughly speaking, the result says that under suitable conditions, any given local algebraic deformations of isolated singularities extend to a global algebraic deformation.

Theorem 4.7 (algebraization). *Let X be a normal projective variety defined over an algebraically closed field \mathbb{k} with only isolated singularities. Assume that*

- (i) $H^2(X, \Theta_{X/\mathbb{k}}) = 0$,
- (ii) *the formal deformation $\mathfrak{X} \rightarrow \mathrm{Spf} \mathfrak{R}$ associated with a pro-representable hull of Def_X is projective, i.e., \mathfrak{X} admits a relatively ample invertible sheaf over $\mathrm{Spf} \mathfrak{R}$.*

Let T be an algebraic Λ -scheme and let $o \in T$ be a \mathbb{k} -rational point. For every singular point $P \in X$, assume that we are given an affine étale neighborhood $(U_{(P)}, P')$ of P in X with a unique singular point and given a deformation $\mathcal{U}_{(P)} \rightarrow T$ of $U_{(P)}$ with reference point o . Then, there exist

- *an étale neighborhood (T', o') of (T, o) , and*
- *a projective morphism $Z \rightarrow T'$ which is a deformation of X with reference point o'*

such that, for any singular point P , (Z, P) and $(\mathcal{U}_{(P)}, P')$ have a common étale neighborhood, i.e., the formal completions of the local rings $\mathcal{O}_{Z,P}$ and $\mathcal{O}_{\mathcal{U}_{(P)},P'}$ are isomorphic to each other ([2, Corollary (2.6)]).

Proof. By Remark 4.3.3 and by Lemma 4.4, for any point $P \in \mathrm{Sing} X$, there exists an algebraic deformation $W_{(P)} \rightarrow T_{(P)}$ of an affine open neighborhood of P over an algebraic Λ -scheme $T_{(P)}$ such that the formal completion of the fiber induces a hull of the deformation functor $\mathrm{Def}_{(X,P)}$. By replacing (T, o) with an étale neighborhood (T', o') , we have a morphism $\varphi_{(P)}: T \rightarrow T_{(P)}$ such that the deformations $\mathcal{U}_{(P)} \rightarrow T$ and $W_{(P)} \times_{T_{(P)}} T$ are equivalent to each other in the sense of [4, Example (4.5)] by the versality there. In particular, $(\mathcal{U}_{(P)}, P')$ and $(W_{(P)} \times_{T_{(P)}} T, P)$ have a common étale neighborhood.

By the assumption (ii) and by Remark 4.3.2, there exist an algebraic Λ -scheme S with a \mathbb{k} -rational point b and a flat projective morphism $W \rightarrow S$ such that

- $W \rightarrow S$ is a deformation of X with the reference point b ,
- the completion of $\mathcal{O}_{S,b}$ is isomorphic to \mathfrak{R} , and
- \mathfrak{X} is isomorphic over $\mathrm{Spf} \mathfrak{R}$ to the formal completion of W along the fiber over b .

Since $\mathrm{Spec} \mathcal{O}_{W,P}$ is a deformation of $\mathrm{Spec} \mathcal{O}_{X,P}$ over S for any $P \in \mathrm{Sing} X$, after replacing (S, b) with an étale neighborhood, we have a morphism $\phi_{(P)}: S \rightarrow T_{(P)}$ such that (W, P) and $(W_{(P)} \times_{T_{(P)}} S, P)$ have a common étale neighborhood as in [4, Example (4.5)].

Let $\phi: S \rightarrow \prod_{P \in \text{Sing } X} T_{(P)}$ be the morphism defined by $\{\phi_{(P)}\}$. Then, ϕ is smooth at b by Theorem 4.6. Hence, we may assume that ϕ is smooth by replacing S with an open neighborhood of b . Let $\varphi: T \rightarrow \prod_{P \in \text{Sing } X} T_{(P)}$ be the morphism defined by $\{\varphi_{(P)}\}$. Since ϕ is smooth, after replacing (T, o) with an étale neighborhood, we have a morphism $\psi: T \rightarrow S$ such that $\psi(o) = b$ and $\varphi = \phi \circ \psi$. Then, the base change $Z = W \times_S T \rightarrow T$ satisfies the expected conditions. \square

Remark 4.7.1. The assumption (ii) above is satisfied if $H^2(X, \mathcal{O}_X) = 0$ (cf. [SGA1, Exp. III, Proposition 7.2]).

5. DEFORMATIONS OF CERTAIN PROJECTIVE SURFACES WITH TORIC SINGULARITIES OF CLASS T

We shall construct some algebraic deformations of a projective normal surface X with toric singularities of class T under extra assumptions. We treat in Theorem 5.2 only deformations over \mathbb{k} , but in Theorem 5.4, deformations over a complete discrete valuation ring with residue field \mathbb{k} . Here, we allow also rational double points on X in Theorem 5.2. As a corollary of Theorem 5.2, in Corollary 5.3, we shall give a correct proof of [40, Theorem 5.16] that any log del Pezzo surface of index two admits a \mathbb{Q} -Gorenstein smoothing to del Pezzo surfaces. Theorem 5.4 is applied to the study of the algebraic fundamental groups of smooth geometric fibers in Corollary 5.5 and Remark 5.5.1.

To begin with, we recall the following well-known result.

Lemma 5.1. *Let X be an affine algebraic \mathbb{k} -variety with a \mathbb{k} -rational point P such that $X \setminus \{P\}$ is non-singular and (X, P) is a local complete intersection singularity, i.e., the local ring $\mathcal{O}_{X,P}$ is a complete intersection. Then, there exist an affine open neighborhood X' , an affine flat morphism $\mathcal{X} \rightarrow T$ over a non-singular curve T , and a \mathbb{k} -rational point $o \in T$ such that*

- (1) $\mathcal{X} \times_T o \simeq X'$,
- (2) $\mathcal{X} \rightarrow T$ is smooth over $T \setminus \{o\}$.

In particular, the singularity (X, P) admits a smoothing.

Proof. The local ring $\mathcal{O}_{X,P}$ is isomorphic to the localization of

$$A = \mathbb{k}[\mathbf{x}_1, \dots, \mathbf{x}_{n+l}]/(f_1, \dots, f_l)$$

at the origin $\{\mathbf{x}_1 = \dots = \mathbf{x}_{n+l} = 0\}$ for a certain regular sequence f_1, \dots, f_l , where $n = \dim X$. Hence, we may assume that $X = \text{Spec } A$ and P is the origin. For $1 \leq i \leq l$, let F_i be the homogeneous polynomial in $\mathbb{k}[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n+l}]$ such that $F_i(1, \mathbf{x}_1, \dots, \mathbf{x}_{n+l}) = f_i$ and $\mathbf{x}_0 \nmid F_i$. Then, the complete intersection closed subscheme $\overline{X} \subset \mathbb{P}_{\mathbb{k}}^{n+l}$ defined by

$\{F_1 = \cdots = F_l = 0\}$ is regarded as the closure of X , i.e., $\overline{X} \cap D_+(\mathbf{x}_0) = X$, where $D_+(\mathbf{x}_0) = \{\mathbf{x}_0 \neq 0\}$. By Bertini's theorem, since \mathbb{k} is algebraically closed, we can take homogeneous polynomials G_1, \dots, G_l with $\deg G_i = \deg F_i$ such that $\{G_i = 0\}$ is non-singular for all $1 \leq i \leq l$ and $\sum_{i=1}^l \{G_i = 0\}$ is a simple normal crossing divisor. For $s, t \in \mathbb{k}$, let $H_i(s, t)$ be the divisor $\{sF_i + tG_i = 0\}$. Then, there is an open neighborhood $T' \subset \mathbb{P}^1$ of $(0 : 1)$ such that, for any closed point $(s : t) \in T'$, $H_i(s, t)$ is non-singular for all $1 \leq i \leq l$ and $\sum_{i=1}^l H_i(s, t)$ is a simple normal crossing divisor. Thus, for $T = T' \cup \{(1 : 0)\}$,

$$\mathcal{X} := \{H_1(s, t) = \cdots = H_l(s, t) = 0\} \cap D_+(\mathbf{x}_0) \rightarrow T$$

is a desired morphism with $o = (1 : 0) \in T$. \square

By [1, Corollary 6], we have:

Corollary 5.1.1. *Any rational Gorenstein surface singularity (rational double point) admits a smoothing.*

The following is our main technical tool for constructing desired surfaces of general type.

Theorem 5.2. *Let \mathbb{k} be an algebraically closed field. Let X be a normal projective surface defined over \mathbb{k} whose singularities are rational double points or toric singularities of class T . Assume that X satisfies the following two conditions:*

- (i) $H^2(X, \Theta_{X/\mathbb{k}}) = 0$.
- (ii) $H^2(X, \mathcal{O}_X) = 0$.

Then, there is a deformation $\mathcal{X} \rightarrow T$ of X over a non-singular algebraic curve T defined over \mathbb{k} with a reference \mathbb{k} -rational point $o \in T$ such that:

- (1) $\mathcal{X} \rightarrow T$ is a projective morphism and it is smooth over $T \setminus \{o\}$.
- (2) \mathcal{X} is normal, $rK_{\mathcal{X}}$ is Cartier, and $\mathcal{O}_{\mathcal{X}}(rK_{\mathcal{X}})|_X \simeq \mathcal{O}_X(rK_X)$ for the Gorenstein index r of X .

In particular, after replacing T with an open neighborhood of o , the following hold for any \mathbb{k} -rational point t of $T \setminus \{o\}$ and the fiber $X_t := \mathcal{X} \times_T t$ over t :

- (3) X_t is a non-singular projective surface defined over \mathbb{k} .
- (4) $\dim H^i(X_t, \mathcal{O}_{X_t}) = \dim H^i(X, \mathcal{O}_X)$ for all $i \geq 0$.
- (5) $K_{X_t}^2 = K_X^2$.
- (6) $H^2(X_t, \Theta_{X_t/\mathbb{k}}) = 0$.
- (7) If K_X (resp. $-K_X$) is ample, then so is K_{X_t} (resp. $-K_{X_t}$).
- (8) If K_X (resp. $-K_X$) is nef and big, then so is K_{X_t} (resp. $-K_{X_t}$).

Proof. In Theorem 5.2, it is enough to consider the deformation theory only in the case where $\Lambda = \mathbb{k}$. First of all, we shall prove (3)–(8) assuming (1) and (2). The assertions (3) and (4) follow from (1), the assumption (ii), and from the upper semi-continuity theorem for the flat morphism $\mathcal{X} \rightarrow T$. Since $(1/2)r^2K_X^2$ is the leading coefficient of the Hilbert polynomial $\chi(X, \mathcal{O}_X(mrK_X))$ with respect to the variable m , (5) follows from (2) and the upper semi-continuity theorem for the flat morphism $\mathcal{X} \rightarrow T$. For (6), let us consider the relative tangent sheaf $\Theta_{\mathcal{X}/T} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{\mathcal{X}/T}^1, \mathcal{O}_X)$, where the canonical homomorphism $\Theta_{\mathcal{X}/T} \otimes_{\mathcal{O}_X} \mathcal{O}_X \rightarrow \Theta_{X/\mathbb{k}}$ is an isomorphism outside $\text{Sing } X$ but another canonical homomorphism $\Theta_{\mathcal{X}/T} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t} \rightarrow \Theta_{X_t/\mathbb{k}}$ is an isomorphism. Then, we have $H^2(X, \Theta_{\mathcal{X}/T} \otimes_{\mathcal{O}_X} \mathcal{O}_X) = 0$ by $H^2(X, \Theta_{X/\mathbb{k}}) = 0$. By the upper semi-continuity theorem applied to the sheaf $\Theta_{\mathcal{X}/T}$ flat over T and by the base change isomorphism, we have the vanishing (6).

The assertion (7) is derived from (2) and [EGA, III, Théorème (4.7.1)]. The last assertion (8) is derived from also a general property. The detail is as follows (cf. [39, Chapter III, §4a, Problem]): Let D be a Cartier divisor on \mathcal{X} such that $D_o := D|_X$ is nef and big. It suffices to show that $D_t = D|_{X_t}$ is also nef and big for any point t in a neighborhood of o . Let H be another Cartier divisor on \mathcal{X} over T . Then, $D_o H_o = D_t H_t$, where $H_t := H|_{X_t}$ by considering the Hilbert polynomial $\chi(X_t, \mathcal{O}_X(m_1 H + m_2 D)|_{X_t})$ of two variables m_1, m_2 . In particular, $D_t^2 = D_o^2 > 0$ and $D_t H_t > 0$ for an ample divisor H_t . Hence, D_t is big. Thus, by replacing T with an affine open neighborhood of o , we have an effective divisor G on \mathcal{X} such that $mD \sim G$ for some $m > 0$. Here, D_t is nef if and only if $D_t C \geq 0$ for any irreducible component C of $G_t = G|_{X_t}$. By eliminating the vertical components of G , we may assume that any irreducible component G_λ of G dominates T . Then, $0 \leq D_o G_{\lambda,o} = D_t G_{\lambda,t}$, where $G_{\lambda,t} = G_\lambda|_{X_t}$. Replacing T by the Stein factorization of $G_\lambda \rightarrow T$ if necessary, we see that D_t is nef for a general point of T .

We shall construct a deformation $\mathcal{X} \rightarrow T$ satisfying (1) and (2) by applying Theorem 4.7. Note that the assumption (i) of Theorem 4.7 is identical to (i) above and the other assumption (ii) of Theorem 4.7 is satisfied by (ii) above (cf. Remark 4.7.1). For a singular point P , we shall construct a deformation $\mathcal{U}_{(P)} \rightarrow T$ of an affine étale neighborhood of (X, P) over a non-singular curve T as follows. If P is a rational double point, then we set $\mathcal{U}_{(P)} \rightarrow T$ to be a deformation obtained in Lemma 5.1 (cf. Corollary 5.1.1), which is a smoothing of (X, P) . Note that $\mathcal{U}_{(P)}$ is normal and Gorenstein. If (X, P) is a toric singularity of class T, then, by Theorem 2.6, there is an affine étale neighborhood $(U_{(P)}, P')$ of (X, P) which is also an étale neighborhood of $(V_{(P)}, \mathbf{0})$ for an affine toric surface $V_{(P)}$ with the closed orbit $\mathbf{0}$. In this case, by applying Theorem 3.8 to $V_{(P)}$ and Lemma 4.4.2 to $(U_{(P)}, P') \rightarrow (V_{(P)}, \mathbf{0})$, we have a deformation $\mathcal{U}_{(P)} \rightarrow T$ of $U_{(P)}$ with a \mathbb{k} -rational reference point o such that

- $\mathcal{U}_{(P)} \rightarrow T$ is smooth over $T \setminus \{o\}$,
- $\mathcal{U}_{(P)}$ is normal, $rK_{\mathcal{U}_{(P)}}$ is Cartier, and

$$\mathcal{O}_{\mathcal{U}_{(P)}}(rK_{\mathcal{U}_{(P)}})|_{U_{(P)}} \simeq \mathcal{O}_{U_{(P)}}(rK_{U_{(P)}})$$

for the Gorenstein index r of (X, P) .

Note that in the construction above, we can take a common non-singular curve T and a reference point o .

Applying Theorem 4.7 to $\mathcal{U}_{(P)} \rightarrow T$ for any $P \in \text{Sing } X$, after replacing (T, o) with an étale neighborhood, we have a deformation $\mathcal{X} \rightarrow T$ of X satisfying the required conditions (1) and (2). Thus, we are done. \square

We shall correct the proof of [40, Theorem 5.16] concerning \mathbb{Q} -Gorenstein smoothings of log del Pezzo surfaces of index two by proving the following stronger assertion as an application of Theorem 5.2.

Corollary 5.3. *Let X be a log del Pezzo surface of index two over an algebraically closed field \mathbb{k} , i.e., X is a normal projective surface, X has only log-terminal singularities, $-K_X$ is an ample non-Cartier divisor, and $2K_X$ is Cartier (cf. [40, Definition 3.2]). Then, there is a projective deformation $\mathcal{X} \rightarrow T$ of X over a non-singular curve T over \mathbb{k} such that*

- $\mathcal{X} \rightarrow T$ is smooth outside X ,
- $2K_{\mathcal{X}}$ is Cartier, and
- any closed fiber of $\mathcal{X} \rightarrow T$ other than X is a del Pezzo surface with $K^2 = K_X^2$.

Proof. A singular point of X is either a rational double point or a singular point of type K_n (cf. [40, Lemma 4.15]), where K_n is just the toric singularity of type $\frac{1}{4n}(1, 2n-1)$: this is of type $T(n, 2, 1)$ (cf. Definition 3.2). We have $H^2(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X(K_X))^\vee = 0$ by the Serre duality theorem, since $-K_X$ is ample. Hence, by Theorem 5.2, it suffices to show $H^2(X, \Theta_{X/\mathbb{k}}) = 0$, or equivalently, $H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\Theta_{X/\mathbb{k}}, \mathcal{O}_X(K_X))) = 0$ by Serre duality. We know that $H^0(X, \mathcal{O}_X(-K_X)) \neq 0$. In fact, for the minimal resolution $\mu: M \rightarrow X$ of singularities, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(-K_X)) &\simeq H^0(M, \mathcal{O}_M(K_M + \mu^*(-2K_X))), \\ H^2(M, \mathcal{O}_M(K_M + \mu^*(-2K_X))) &\simeq H^0(M, \mu^*\mathcal{O}_X(2K_X))^\vee = 0, \quad \text{and} \\ \dim H^0(M, \mathcal{O}_M(K_M + \mu^*(-2K_X))) &\geq \chi(M, \mathcal{O}_M(K_M + \mu^*(-2K_X))) \\ &= \frac{1}{2}(K_M + \mu^*(-2K_X))\mu^*(-2K_X) + 1 = K_X^2 + 1 > 0. \end{aligned}$$

Taking a non-zero section of $\mathcal{O}_X(-K_X)$, we obtain an injection $\mathcal{O}_X(K_X) \hookrightarrow \mathcal{O}_X$, and hence an injection

$$\mathcal{H}om_{\mathcal{O}_X}(\Theta_{X/\mathbb{k}}, \mathcal{O}_X(K_X)) \hookrightarrow \mathcal{H}om_{\mathcal{O}_X}(\Theta_{X/\mathbb{k}}, \mathcal{O}_X) \simeq (\Omega_{X/\mathbb{k}}^1)^{\vee\vee},$$

where the right sheaf is isomorphic to $\mu_*\Omega_{M/\mathbb{k}}^1$ by Proposition 2.11.(3). Since M is rational,

$$H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\Theta_{X/\mathbb{k}}, \mathcal{O}_X(K_X))) \subset H^0(M, \Omega_{M/\mathbb{k}}^1) = 0.$$

Thus, we are done. \square

Theorem 5.4. *Let Λ be a complete discrete valuation ring with an algebraically closed residue field \mathbb{k} . Let $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$ be a flat projective morphism satisfying the following two conditions:*

- (i) *The closed fiber $X = X_{\mathbb{k}}$ is a normal projective surface with only toric singularities of class T satisfying $H^2(X, \Theta_{X/\mathbb{k}}) = H^2(X, \mathcal{O}_X) = 0$.*
- (ii) *For any singular point P of X , let $(n_{(P)}, q_{(P)})$ be the type of the toric singularity. Then, there exist an affine neighborhood $Y_{(P)}$ of P in X_Λ and two prime divisors $B_{1(P)}, B_{2(P)}$ on $Y_{(P)}$ containing P such that $(Y_{(P)}, B_{1(P)}, B_{2(P)})$ satisfies the condition $C(n_{(P)}, q_{(P)})'$ over $\operatorname{Spec} \Lambda$ (cf. Definition 2.1).*

Then, there exist an algebraic deformation $Z \rightarrow T$ of X over an algebraic smooth Λ -scheme T of relative dimension one, a \mathbb{k} -rational point $o \in T$, and a section $\sigma: \operatorname{Spec} \Lambda \rightarrow T$ such that:

- (1) $\sigma(\mathfrak{m}_\Lambda) = o$ and $Z \times_{T, \sigma} \operatorname{Spec} \Lambda \simeq X_\Lambda$.
- (2) Z is smooth over $T \setminus \sigma(\operatorname{Spec} \Lambda)$.
- (3) Z is normal, rK_Z is Cartier, and $\mathcal{O}_Z(rK_Z)|_{X_\Lambda} \simeq \mathcal{O}_{X_\Lambda}(rK_{X_\Lambda})$ for the Gorenstein index r of X .

Proof. By Theorems 2.7, 3.8 and by Lemma 4.4.2, we have an algebraic smooth Λ -scheme T of relative dimension one, a \mathbb{k} -rational reference point $o \in T$, a section $\sigma: \operatorname{Spec} \Lambda \rightarrow T$ with $\sigma(\mathfrak{m}_\Lambda) = o$, and a flat family $V_{(P)} \rightarrow T$ of normal affine surfaces for any singular point P of X such that

- (V1) $V_{(P)} \times_{T, \sigma} \operatorname{Spec} \Lambda \simeq Y_{(P)}$,
- (V2) $V_{(P)} \rightarrow T$ is smooth over $T \setminus \sigma(\operatorname{Spec} \Lambda)$,
- (V3) $V_{(P)}$ is normal, $r_P K_{V_{(P)}}$ is Cartier with $\mathcal{O}_{V_{(P)}}(r_P K_{V_{(P)}})|_{Y_{(P)}} \simeq \mathcal{O}_{Y_{(P)}}(r_P K_{Y_{(P)}})$ for the Gorenstein index r_P of the toric singularity $P \in X$.

In fact, $(Y_{(P)}, P)$ is an étale neighborhood of an affine toric surface at the closed orbit by Theorem 2.7, and the toric surface admits a deformation satisfying conditions (1)–(3) of Theorem 3.8 since it is of class T , and finally by Lemma 4.4.2, we can lift the deformation to that of $Y_{(P)}$. Here, the section in Theorem 3.8 induces the section σ , since Λ is a complete discrete valuation ring.

Let $\mathfrak{X} \rightarrow \operatorname{Spf} \mathfrak{R}$ be the formal deformation associated with a hull (\mathfrak{R}, ξ) of Def_X . By the assumption (i), we can find an algebraization $W \rightarrow S$ of $\mathfrak{X} \rightarrow \operatorname{Spf} \mathfrak{R}$ as in the proof

of Theorem 4.7 by Remark 4.7.1. Here, $W \rightarrow S$ is a projective flat morphism over an algebraic Λ -scheme S , the fiber over a \mathbb{k} -rational point $b \in S$ is isomorphic to X , the completion of $\mathcal{O}_{S,b}$ is isomorphic to \mathfrak{R} , and the formal completion of W along X is isomorphic to \mathfrak{X} . Since $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$ is a deformation of X with reference point \mathfrak{m}_Λ , we have a surjection $\mathfrak{R} \rightarrow \Lambda$ such that $\mathfrak{X} \times_{\operatorname{Spf} \mathfrak{R}} \operatorname{Spf} \Lambda$ is isomorphic to the formal completion of X_Λ along X . Hence, for the induced section $\sigma_1: \operatorname{Spec} \Lambda \rightarrow \operatorname{Spec} \mathfrak{R} \rightarrow \operatorname{Spec} \mathcal{O}_{S,b} \rightarrow S$, we have an isomorphism $W \times_{S, \sigma_1} \operatorname{Spec} \Lambda \simeq X_\Lambda$ by [EGA, III, Théorème (5.4.1)].

Let $W_{(P)} \rightarrow T_{(P)}$ be an algebraization of the formal deformation associated with a hull of $\operatorname{Def}_{(X,P)}$ as in the proof of Theorem 4.7. After replacing (S, b) with an étale neighborhood and replacing (T, o) with an étale neighborhood, we have morphisms $\phi: S \rightarrow \prod_{P \in \operatorname{Sing} X} T_{(P)}$ and $\varphi: T \rightarrow \prod_{P \in \operatorname{Sing} X} T_{(P)}$ such that $\varphi \circ \sigma = \phi \circ \sigma_1$ (cf. (V1)) and that, for any $P \in \operatorname{Sing} X$,

- (W, P) and $(W_{(P)} \times_{T_{(P)}} S, P)$ have a common étale neighborhood,
- $(V_{(P)}, P)$ and $(W_{(P)} \times_{T_{(P)}} T, P)$ have a common étale neighborhood.

Let $S_T \rightarrow T$ be the base change of ϕ by φ . By replacing T with an open neighborhood of o , we may assume that $S_T \rightarrow T$ is smooth, since ϕ is smooth at b by Theorem 4.6. Now, we have a section $\sigma' = (\sigma, \sigma_1): \operatorname{Spec} \Lambda \rightarrow S_T$. Then, after replacing (T, o) with an étale neighborhood, we have a section $\psi: T \rightarrow S_T$ such that $\psi \circ \sigma = \sigma'$. In fact, S_T is étale over $\mathbb{A}_T^k := \mathbb{A}_\Lambda^k \times_{\operatorname{Spec} \Lambda} T$ for some k , and σ' induces a section of \mathbb{A}_Λ^k over $\operatorname{Spec} \Lambda$. Here, we may assume that the section is defined by $\mathfrak{t}_1 = \cdots = \mathfrak{t}_k = 0$ for $\mathbb{A}_\Lambda^k = \operatorname{Spec} \Lambda[\mathfrak{t}_1, \dots, \mathfrak{t}_k]$. The closed subscheme of \mathbb{A}_T^k defined by $\mathfrak{t}_1 = \cdots = \mathfrak{t}_k = 0$ is isomorphic to T . Hence a connected component of the pullback of the closed subscheme by $S_T \rightarrow \mathbb{A}_T^k$ gives a desired étale neighborhood. Let $Z \rightarrow T$ be the base change of $W \rightarrow S$ by $T \rightarrow S_T \rightarrow S$. Then, $Z \rightarrow T$ is a deformation of X with reference point o satisfying the condition (1), and (Z, P) and $(V_{(P)}, P)$ have a common étale neighborhood for any $P \in \operatorname{Sing} X$. Hence, the other conditions (2) and (3) are derived from (V2) and (V3), and we have finished the proof. \square

Corollary 5.5 (Fundamental group). *Let $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$ be the flat projective morphism in Theorem 5.4 satisfying the two conditions (i) and (ii). Assume that the field of fractions of Λ is of characteristic zero. Let X be the closed fiber of $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$. Let \mathbb{K} be an algebraically closed field containing Λ and let $X_{\mathbb{K}}$ be the fiber product $X_\Lambda \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \mathbb{K}$. Then, $H^2(X_{\mathbb{K}}, \mathcal{O}_{X_{\mathbb{K}}}) = H^2(X_{\mathbb{K}}, \Theta_{X_{\mathbb{K}}/\mathbb{K}}) = 0$. Moreover, there exist a deformation $\mathcal{X} \rightarrow C$ of X over a non-singular algebraic curve C defined over \mathbb{k} , and a deformation $\mathcal{Y} \rightarrow D$ of $X_{\mathbb{K}}$ over a non-singular algebraic curve D defined over \mathbb{K} such that $\mathcal{X} \rightarrow C$ and $\mathcal{Y} \rightarrow D$ satisfy the conditions corresponding to (1) and (2) of Theorem 5.2 and that there is a*

surjection

$$\pi_1^{\text{alg}}(Y_d) \rightarrow \pi_1^{\text{alg}}(X_c)$$

of algebraic fundamental groups for any smooth fibers X_c and Y_d of $\mathcal{X} \rightarrow C$ and $\mathcal{Y} \rightarrow D$ over closed points $c \in C$ and $d \in D$, respectively.

Proof. First, we shall show: $H^2(X_{\mathbb{K}}, \mathcal{O}_{X_{\mathbb{K}}}) = H^2(X_{\mathbb{K}}, \Theta_{X_{\mathbb{K}}/\mathbb{K}}) = 0$. Since $H^2(X, \mathcal{O}_X) = 0$, we have $H^2(X_{\Lambda}, \mathcal{O}_{X_{\Lambda}}) = 0$ by the upper semi-continuity theorem for the flat morphism $X_{\Lambda} \rightarrow \text{Spec } \Lambda$, and $H^2(X_{\mathbb{K}}, \mathcal{O}_{X_{\mathbb{K}}}) = 0$ by the flat base change isomorphism $H^2(X_{\Lambda}, \mathcal{O}_{X_{\Lambda}}) \otimes_{\Lambda} \mathbb{K} \simeq H^2(X_{\mathbb{K}}, \mathcal{O}_{X_{\mathbb{K}}})$. The vanishing of $H^2(X_{\mathbb{K}}, \Theta_{X_{\mathbb{K}}/\mathbb{K}})$ is shown as in the proof of Theorem 5.2.(6): For the relative tangent sheaf $\Theta_{X_{\Lambda}/\Lambda} := \mathcal{H}om_{\mathcal{O}_{X_{\Lambda}}}(\Omega_{X_{\Lambda}/\Lambda}^1, \mathcal{O}_{X_{\Lambda}})$, the canonical homomorphism $\Theta_{X_{\Lambda}/\Lambda} \otimes_{\mathcal{O}_{X_{\Lambda}}} \mathcal{O}_X \rightarrow \Theta_{X/\mathbb{K}}$ is an isomorphism outside $\text{Sing } X$ but another canonical homomorphism $\Theta_{X_{\Lambda}/\Lambda} \otimes_{\Lambda} \mathbb{K} \rightarrow \Theta_{X_{\mathbb{K}}/\mathbb{K}}$ is an isomorphism. Thus, we have $H^2(X, \Theta_{X_{\Lambda}/\Lambda} \otimes_{\mathcal{O}_{X_{\Lambda}}} \mathcal{O}_X) = 0$ by $H^2(X, \Theta_{X/\mathbb{K}}) = 0$. By the upper semi-continuity theorem applied to the sheaf $\Theta_{X_{\Lambda}/\Lambda}$ flat over Λ and by the base change isomorphism, we have the vanishing $H^2(X_{\mathbb{K}}, \Theta_{X_{\mathbb{K}}/\mathbb{K}}) = 0$.

Second, we shall define $\mathcal{X} \rightarrow C$ and $\mathcal{Y} \rightarrow D$. Let $Z \rightarrow T$ be the deformation of X obtained in Theorem 5.4. By the surjection $\Lambda \twoheadrightarrow \mathbb{K}$ and the injection $\Lambda \hookrightarrow \mathbb{K}$, we define

$$\begin{aligned} \mathcal{X} &:= Z \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{K} \rightarrow C := T \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{K}, \quad \text{and} \\ \mathcal{Y} &:= Z \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{K} \rightarrow D := C \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{K}. \end{aligned}$$

Then, $\mathcal{X} \rightarrow C$ is a deformation of X with reference point $o = C \cap \sigma(\text{Spec } \Lambda)$, and $\mathcal{Y} \rightarrow D$ is a deformation of $X_{\mathbb{K}}$ with the reference point $o_{\mathbb{K}} := D \times_Z \sigma(\text{Spec } \Lambda)$. Moreover, these deformations satisfy the conditions corresponding to (1) and (2) of Theorem 5.2.

Finally, we shall compare several algebraic fundamental groups using results in [SGA1, Exp. X] concerning with Grothendieck's specialization theorem [SGA1, Exp. X, Corollaire 2.4, Théorème 3.8]. Let \mathbb{K}_1 be the algebraic closure of the function field $\mathbb{K}(C)$ of C and set $\mathcal{X}_{\mathbb{K}_1}$ to be the fiber product $\mathcal{X} \times_C \text{Spec } \mathbb{K}_1$. Then, for any smooth closed fiber X_c of $\mathcal{X} \rightarrow C$, we have a surjection

$$(V-1) \quad \pi_1^{\text{alg}}(\mathcal{X}_{\mathbb{K}_1}) \twoheadrightarrow \pi_1^{\text{alg}}(X_c)$$

by [SGA1, Exp. X, Corollaire 2.4, Théorème 3.8].

Let \mathbb{K}_1 be an algebraically closed field containing the function field $\mathbb{K}(D)$ of D , and let $\mathcal{Y}_{\mathbb{K}_1}$ be the fiber product $\mathcal{Y} \times_D \text{Spec } \mathbb{K}_1$. The geometric generic points $\text{Spec } \mathbb{K}_1 \rightarrow D \rightarrow T$ and $\text{Spec } \mathbb{K}_1 \rightarrow C \rightarrow T$ are lying on the open subset $T \setminus \sigma(\text{Spec } \Lambda)$, and the corresponding geometric fibers of $Z \rightarrow T$ are $\mathcal{Y}_{\mathbb{K}_1}$ and $\mathcal{X}_{\mathbb{K}_1}$, respectively. By [SGA1, Exp. X, Corollaire 2.4, Théorème 3.8], we have a surjection

$$(V-2) \quad \pi_1^{\text{alg}}(\mathcal{Y}_{\mathbb{K}_1}) \twoheadrightarrow \pi_1^{\text{alg}}(\mathcal{X}_{\mathbb{K}_1})$$

Since $\text{char}(\mathbb{K}) = 0$, by [SGA1, Exp. X, Théorème 3.8, Corollaire 3.9], we have an isomorphism

$$(V-3) \quad \pi_1^{\text{alg}}(\mathcal{Y}_{\mathbb{K}_1}) \simeq \pi_1^{\text{alg}}(Y_d)$$

for any smooth closed fiber Y_d of $\mathcal{Y} \rightarrow D$. Thus, we have a desired surjection $\pi_1^{\text{alg}}(Y_d) \twoheadrightarrow \pi_1^{\text{alg}}(X_c)$ from (V-1)–(V-3). \square

Remark 5.5.1. The calculation of $\pi_1^{\text{alg}}(Y_d)$ in Corollary 5.5 is reduced to the case over the complex number field \mathbb{C} , as follows. Let \mathbb{K} , $X_{\mathbb{K}}$, and $\mathcal{Y} \rightarrow D$ be as in Corollary 5.5. Here, $\mathcal{Y} \rightarrow D$ is a deformation of $X_{\mathbb{K}}$ with a reference \mathbb{K} -rational point $b := o_{\mathbb{K}} \in D$ as in the proof of Corollary 5.5. Then, there is a finitely generated field \mathbb{K}_0 over the field \mathbb{Q} of rational numbers such that $X_{\mathbb{K}}$, $\mathcal{Y} \rightarrow D$, $o_{\mathbb{K}} \in D$, $d \in D \setminus \{b\}$, and every point of $\text{Sing } X_{\mathbb{K}} = \{P_1, \dots, P_k\}$ descend to over \mathbb{K}_0 . Namely, there exist algebraic schemes X_0 and D_0 over $\text{Spec } \mathbb{K}_0$, \mathbb{K}_0 -rational points b_0 and d_0 of D_0 , \mathbb{K} -rational points $P_{1,0}, \dots, P_{k,0}$ of X_0 , and a morphism $\mathcal{Y}_0 \rightarrow D_0$ such that

$$\begin{aligned} X_{\mathbb{K}} &\simeq X_0 \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K}, & D &\simeq D_0 \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K}, \\ (b \in D) &\simeq (b_0 \in D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K}, & (d \in D) &\simeq (d_0 \in D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K}, \\ (P_i \in X) &\simeq (P_{i,0} \in X_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K} & (1 \leq i \leq k), \\ (\mathcal{Y} \rightarrow D) &\simeq (\mathcal{Y}_0 \rightarrow D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{K}, \end{aligned}$$

with the following properties (cf. [EGA, IV, Propositions (2.5.1), (2.7.1), (6.7.4), Corollaire (2.7.2)]):

- X_0 is a normal projective integral \mathbb{K}_0 -scheme, and $X_0 \setminus \{P_{i,0}\}_{1 \leq i \leq k}$ is smooth over $\text{Spec } \mathbb{K}_0$.
- D_0 is a smooth algebraic \mathbb{K}_0 -scheme.
- \mathcal{Y}_0 is normal and integral.
- $\mathcal{Y}_0 \rightarrow D_0$ is a projective flat morphism whose fiber over b_0 is identified with X_0 .
- $\mathcal{Y}_0 \rightarrow D_0$ is smooth on $\mathcal{Y}_0^\circ := \mathcal{Y}_0 \setminus \{P_{i,0}\}_{1 \leq i \leq k}$.

Moreover, $rK_{\mathcal{Y}_0/D_0}$ is Cartier for the relative canonical divisor $K_{\mathcal{Y}_0/D_0}$ and for the index r of X , and there is an isomorphism

$$\mathcal{O}_{\mathcal{Y}_0}(rK_{\mathcal{Y}_0/D_0}) \simeq j_*(\omega_{\mathcal{Y}_0^\circ/D_0}^{\otimes r})$$

for the relative dualizing sheaf $\omega_{\mathcal{Y}_0^\circ/D_0}$ where $j: \mathcal{Y}_0^\circ \hookrightarrow \mathcal{Y}_0$ denotes the open immersion. These properties are also derived from the corresponding properties on $\mathcal{Y} \rightarrow D$. Note that the fiber Y_d of $\mathcal{Y} \rightarrow D$ over d is the base change of the fiber Y_{d_0} of $\mathcal{Y}_0 \rightarrow D_0$ over d_0

by $\text{Spec } \mathbb{K} \rightarrow \text{Spec } \mathbb{K}_0$. Taking a field extension $\mathbb{K}_0 \subset \mathbb{C}$, we set

$$\begin{aligned} X_{\mathbb{C}} &:= X_0 \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}, & D_{\mathbb{C}} &:= D_0 \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}, \\ (b_{\mathbb{C}} \in D_{\mathbb{C}}) &:= (b_0 \in D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}, & (d_{\mathbb{C}} \in D_{\mathbb{C}}) &:= (d_0 \in D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}, \\ (P_{i,\mathbb{C}} \in D_{\mathbb{C}}) &:= (P_{i,0} \in D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}, & (1 \leq i \leq k), \\ (\mathcal{Y}_{\mathbb{C}} \rightarrow D_{\mathbb{C}}) &:= (\mathcal{Y}_0 \rightarrow D_0) \times_{\text{Spec } \mathbb{K}_0} \text{Spec } \mathbb{C}. \end{aligned}$$

By considering also the descent of the minimal resolution of singularities of X , we may assume that $X_{\mathbb{C}}$ has only toric singularities of class T. In fact, the exceptional locus over $P_{i,\mathbb{C}}$ of the minimal resolution of $X_{\mathbb{C}}$ is the linear chain of rational curves with the same self-intersection numbers as that for P_i ; thus $(X_{\mathbb{C}}, P_{i,\mathbb{C}})$ is a toric singularity by Theorem 2.6. Then, $\mathcal{Y}_{\mathbb{C}} \rightarrow D_{\mathbb{C}}$ is a \mathbb{Q} -Gorenstein smoothing of $X_{\mathbb{C}}$ in the sense of [26, Section 3] (cf. [26, Corollary 3.6]), since $\mathcal{Y}_{\mathbb{C}}$ is \mathbb{Q} -Gorenstein. For the smooth fiber $Y_{\mathbb{C},d_{\mathbb{C}}}$ of $\mathcal{Y}_{\mathbb{C}} \rightarrow D_{\mathbb{C}}$ over $d_{\mathbb{C}}$, we have an isomorphism

$$\pi_1^{\text{alg}}(Y_d) \simeq \pi_1^{\text{alg}}(Y_{\mathbb{C},d_{\mathbb{C}}})$$

by [SGA1, Exp. X, Corollaire 1.8].

6. SIMPLY CONNECTED SURFACES OF GENERAL TYPE WITH $p_g = q = 0$

We apply the results in Sections 2–5 to construct algebraically simply connected surfaces \mathbb{S} of general type with $p_g = q = 0$ and $1 \leq K^2 \leq 4$ which is defined over the given algebraically closed field \mathbb{k} , where

$$p_g = p_g(\mathbb{S}) = \dim H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(K_{\mathbb{S}})) = \dim H^2(\mathbb{S}, \mathcal{O}_{\mathbb{S}}), \quad q = q(\mathbb{S}) = \dim H^1(\mathbb{S}, \mathcal{O}_{\mathbb{S}}),$$

and $K = K_{\mathbb{S}}$ denotes the canonical divisor. An outline of our method is as follows. We first construct a normal projective rational surface X with only toric singularities of class T satisfying the following conditions:

- K_X is nef and big (or ample),
- K_X^2 equals the given number $K^2 > 0$,
- $H^2(X, \Theta_X) = 0$.

Note that $H^i(X, \mathcal{O}_X) = 0$ for any $i > 0$ since X is rational and has only rational singularities. For the construction of X , we follow the method used in [33], [42], and [43], which is however considered over the field \mathbb{C} of complex numbers. By Theorem 5.2, we have a projective deformation $\mathcal{X} \rightarrow T$ of X over a non-singular curve T defined over \mathbb{k} satisfying the conditions (1) and (2) of Theorem 5.2. Here, a general closed smooth fiber X_t of $\mathcal{X} \rightarrow T$ is a non-singular projective surface \mathbb{S} having the following properties:

- $q(\mathbb{S}) = p_g(\mathbb{S}) = 0$ and $\chi(\mathbb{S}, \mathcal{O}_{\mathbb{S}}) = 1$.

- \mathbb{S} is a minimal surface of general type with $K_{\mathbb{S}}^2 = K^2$.
- $H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = 0$. In particular, \mathbb{S} is liftable to characteristic zero.

For the second condition above, note that K_{X_t} is ample if K_X is. If X_t is algebraically simply connected, then this is one of the surfaces what we want to get. In order to construct a simply connected one, we select the deformation $\mathcal{X} \rightarrow T$ by considering a lifting problem to characteristic zero. Namely, we construct X as the closed fiber of a flat family $X_{\Lambda} \rightarrow \operatorname{Spec} \Lambda$ for a complete discrete valuation ring Λ of mixed characteristic with the residue field \mathbb{k} satisfying the assumptions (1) and (2) of Theorem 5.4. Then, by Corollary 5.5, we have a deformation $\mathcal{X} \rightarrow T$ satisfying the conditions (1) and (2) of Theorem 5.2 in which $\pi_1^{\text{alg}}(X_t)$ is dominated by π_1^{alg} of a \mathbb{Q} -Gorenstein smoothing of a geometric generic fiber of $X_{\Lambda} \rightarrow \operatorname{Spec} \Lambda$. Looking at the construction of X_{Λ} , we shall prove the simply connectedness of the \mathbb{Q} -Gorenstein smoothing of a geometric generic fiber from the argument used in the proof of [33, Theorem 3.1], [42, Theorem 3.1], and [43, Proposition 2.1]. The argument shows especially that, when $\mathbb{k} = \mathbb{C}$, $X \setminus \operatorname{Sing} X$ is topologically simply connected. In this way, the new X_t is shown to be algebraically simply connected, and we have a desired surface.

Remark 6.1. In the construction above, the number K^2 must be between 1 and 4. In fact, we may assume that $\operatorname{char}(\mathbb{k}) = 0$, and in this case, we have $H^0(X_t, \Theta_{X_t/\mathbb{k}}) = 0$, since the automorphism group of the surface X_t of general type is finite. Thus, by Riemann–Roch,

$$\dim H^1(X_t, \Theta_{X_t/\mathbb{k}}) = -\chi(X_t, \Theta_{X_t/\mathbb{k}}) = 10 - 2K^2,$$

where we use $\chi(X_t, \mathcal{O}_{X_t}) = 1$, $K_{X_t}^2 = K^2$, and $H^2(X_t, \Theta_{X_t/\mathbb{k}}) = 0$. Hence, $1 \leq K^2 \leq 5$. Moreover, our X_t is a fiber of a \mathbb{Q} -Gorenstein smoothing of a rational surface with only cyclic quotient singularities of class T. Hence, X_t has a non-trivial deformation by the existence of the coarse moduli of surfaces of general type (cf. [16, Theorem 1.3], [26, Corollary 5.7]). Thus, $\dim H^1(X_t, \Theta_{X_t/\mathbb{k}}) > 0$ and $K^2 \leq 4$.

We shall explain the construction of X and $X_{\Lambda} \rightarrow \operatorname{Spec} \Lambda$ from suitable cubic pencils on \mathbb{P}^2 step by step in Section 6. We also give sufficient conditions for the surface X to be a desired one. Explicit examples of the cubic pencils are given in Section 7, and the Main Theorem is proved using these examples.

Let us fix a complete discrete valuation ring Λ of mixed characteristic with the residue field \mathbb{k} . Let K^2 be a given positive integer.

Step 1. We first take two cubic homogeneous polynomials ϕ_0, ϕ_{∞} from $\mathbb{Z}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, and let Φ_0 and Φ_{∞} be the divisors of zeros of ϕ_0 and ϕ_{∞} , respectively, on $\mathbb{P}_{\mathbb{k}}^2 = \operatorname{Proj} \mathbb{k}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. Let Φ be the cubic pencil defined by Φ_0 and Φ_{∞} . For $c \in \mathbb{k}$, let Φ_c be the divisor of zeros of $\phi_0 + c\phi_{\infty}$. We here require the following conditions on Φ :

- (C1) The base locus of Φ , i.e. $\Phi_0 \cap \Phi_\infty$, is a finite set.
- (C2) There exist at least two values of $c \in \mathbb{k} \setminus \{0\}$ such that Φ_c is singular. Moreover, if Φ_c is singular for $c \neq 0$, then it is a nodal rational curve.

Let $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ be the elimination of the base locus of Φ which is a succession of blowings up at points. Then, we have a minimal elliptic fibration $\pi: Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ with a section such that $\mathcal{O}_Y(-K_Y) \simeq \pi^*\mathcal{O}(1)$.

Lemma 6.2. *In the situation of Step 1, there is a flat morphism $\pi_\Lambda: Y_\Lambda \rightarrow \mathbb{P}_\Lambda^1$ such that π is the base change of π_Λ by the closed immersion $\mathbb{P}_{\mathbb{k}}^1 \rightarrow \mathbb{P}_\Lambda^1$. Let c_1, c_2, \dots, c_n be the values $c \in \mathbb{k}$ such that Φ_c is singular. Then, c_i are regarded as elements of Λ , and hence π_Λ is a smooth elliptic fibration outside the sections of $\mathbb{P}_\Lambda^1 \rightarrow \text{Spec } \Lambda$ defined by $(u : v) = (1 : c_i)$ for $1 \leq i \leq n$ and $(u : v) = (0 : 1)$, where $(u : v)$ is a homogeneous coordinate of \mathbb{P}^1 .*

Proof. By construction, Φ_0 and Φ_∞ are defined over \mathbb{Z} , and the each center of the blowing up between Y and $\mathbb{P}_{\mathbb{k}}^2$ is a point whose coordinates are algebraic over \mathbb{Z} . Thus, the point is defined over the Henselian local ring Λ . Hence, $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ extends to a birational morphism $Y_\Lambda \rightarrow \mathbb{P}_\Lambda^2$ which is a succession of blowings up along centers over $\text{Spec } \Lambda$. Moreover, by the pencil over Λ , we have an elliptic fibration $\pi_\Lambda: Y_\Lambda \rightarrow \mathbb{P}_\Lambda^1$ extending π . The elements c_1, \dots, c_n can be regarded as elements of Λ , since Λ is Henselian, and π_Λ is smooth outside the sections above. \square

Step 2. For the elliptic fibration π in *Step 1*, after choosing two values $c_1, c_2 \in \mathbb{k} \setminus \{0\}$ such that Φ_{c_1} and Φ_{c_2} are singular, we define \bar{F}_1 and \bar{F}_2 to be the proper transforms of Φ_{c_1} and Φ_{c_2} in Y , respectively. Note that, for $i = 1, 2$, \bar{F}_i is the fiber over the point $(1 : c_i) \in \mathbb{P}_{\mathbb{k}}^1$, and it is a singular fiber of type I_1 in Kodaira's notation (cf. [25, Theorem 6.2]). We define $\bar{F} := \bar{F}_1 + \bar{F}_2$. Let \bar{G}^+ be the union of all the (-2) -curves on Y . Note that every (-2) -curve on Y is an irreducible component of a reducible fiber by $\mathcal{O}_Y(-K_Y) \simeq \pi^*\mathcal{O}(1)$, and thus it is an irreducible component of the total transform of $\Phi_0 + \Phi_\infty$ in Y (cf. (C2)). We choose (-2) -curves $\bar{G}_1, \bar{G}_2, \dots, \bar{G}_k$ on Y and set $\bar{G} := \sum_{i=1}^k \bar{G}_i$. Moreover, we choose horizontal curves $\bar{S}_1, \bar{S}_2, \dots, \bar{S}_l$ with respect to π such that each \bar{S}_j is either exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ or its image in $\mathbb{P}_{\mathbb{k}}^2$ is a line defined over the ring of algebraic integers. We set $\bar{S} := \sum_{j=1}^l \bar{S}_j$, and also

$$\bar{B} := \bar{F} + \bar{G} + \bar{S} \quad \text{and} \quad \bar{B}^+ := \bar{F} + \bar{G}^+ + \bar{S}.$$

Note that, by construction and by Lemma 6.2, any irreducible component of \bar{B}^+ is realized as the closed fiber of a prime divisor of Y_Λ over $\text{Spec } \Lambda$. We require the following conditions on Y , \bar{B}^+ , and \bar{G} :

(C3) \bar{B}^+ is a simple normal crossing divisor outside the nodes of \bar{F} .

(C4) $\mathcal{O}_Y(\bar{G}_1), \dots, \mathcal{O}_Y(\bar{G}_k)$, and $\pi^*\mathcal{O}(1)$ are linearly independent in $\text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{k}$.

We do not require the following additional conditions (A1) and (A2) which are however useful to check the ampleness of K_X (cf. Proposition 6.6.(4) below):

(A1) The open set $Y \setminus (\bar{S} \cup \bar{G})$ is affine.

(A2) The open set $Y \setminus (\bar{S} \cup \bar{G})$ does not contain any (-2) -curve.

Lemma 6.3. *In the situation of Step 2, the condition (C4) is satisfied if*

$$\det(\bar{G}_i \bar{G}_j) \not\equiv 0 \pmod{\text{char}(\mathbb{k})}.$$

Proof. For $p := \text{char}(\mathbb{k})$, suppose that

$$\sum_{i=1}^k c_i \bar{G}_i + c \bar{F}_1 \sim p H$$

for some integers c_i and c , and for the fiber \bar{F}_1 of π and a Cartier divisor H . Then, $c_i \equiv 0 \pmod{p}$ for any $1 \leq i \leq k$ by $\det(\bar{G}_i \bar{G}_j) \not\equiv 0 \pmod{p}$ and by $\bar{F}_1 \bar{G}_i = 0$. Moreover, $c \equiv 0 \pmod{p}$, since $\bar{F}_1 \Sigma = 1$ for a section Σ of π . Thus, (C4) is satisfied. \square

Lemma 6.4. *In the situation of Step 2,*

$$H^0(Y, \Omega_{Y/\mathbb{k}}^1(\log(\bar{G} + \bar{F}_2))) = 0.$$

Proof. Since $\bar{G} + \bar{F}_2$ is normal crossing (cf. (C3)), we can consider the exact sequence

$$(VI-1) \quad 0 \rightarrow \Omega_{Y/\mathbb{k}}^1 \rightarrow \Omega_{Y/\mathbb{k}}^1(\log(\bar{G} + \bar{F}_2)) \rightarrow \bigoplus_{i=1}^k \mathcal{O}_{\bar{G}_i} \oplus \mathcal{O}_{F_2} \rightarrow 0,$$

where F_2 is the normalization of \bar{F}_2 . Let $f_2 \in H^1(Y, \Omega_{Y/\mathbb{k}}^1)$ be the image of $1 \in H^0(\mathcal{O}_{F_2})$ by the connecting homomorphism of the long exact sequence of cohomology groups associated with (VI-1). Similarly, let $g_i \in H^1(Y, \Omega_{Y/\mathbb{k}}^1)$ be the image of $1 \in H^0(\mathcal{O}_{G_i})$ for $1 \leq i \leq k$. Then, f_2 and g_i can be regarded as the first Chern classes (with respect to “dlog”) of the invertible sheaves $\mathcal{O}_Y(\bar{F}_2)$ and $\mathcal{O}_Y(\bar{G}_i)$, respectively. In fact, we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_Y^\times & \longrightarrow & \mathcal{O}_Y(*(\bar{G} + \bar{F}_2))^\times & \longrightarrow & \mathcal{H}_{\bar{G} + \bar{F}_2}^0(\text{Div}_Y) \longrightarrow 0 \\ & & \text{dlog} \downarrow & & \text{dlog} \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{Y/\mathbb{k}}^1 & \longrightarrow & \Omega_{Y/\mathbb{k}}^1(\log(\bar{G} + \bar{F}_2)) & \longrightarrow & \bigoplus_{i=1}^k \mathcal{O}_{\bar{G}_i} \oplus \mathcal{O}_{F_2} \longrightarrow 0 \end{array}$$

of exact sequence, where $\text{dlog}(u) = u^{-1}du$ for a rational function u , and for an effective divisor R ,

- $\mathcal{O}_Y(*R)^\times$ stands for the sheaf of invertible rational functions on Y regular on the open subset $Y \setminus \text{Supp } R$, and
- $\mathcal{H}_R^0(\text{Div}_Y)$ stands for the sheaf of Cartier divisors on Y supported on $\text{Supp } R$.

Note that, as global sections of $\mathcal{H}_{\bar{G}+\bar{F}_2}^0(\mathcal{D}iv_Y)$, \bar{F}_2 and \bar{G}_i are mapped to $1 \in \mathcal{O}_{F_2}$ and $1 \in \mathcal{O}_{\bar{G}_i}$, respectively, by the right vertical homomorphism. The first Chern class map $c: \text{Pic}(Y) = H^1(Y, \mathcal{O}_Y^\times) \rightarrow H^1(Y, \Omega_{Y/\mathbb{k}}^1)$ is induced by the left vertical homomorphism: dlog . Therefore, $f_2 = c(\mathcal{O}_Y(\bar{F}_2))$ and $g_i = c(\mathcal{O}_Y(\bar{G}_i))$. For the natural bilinear form

$$\langle \ , \ \rangle: H^1(Y, \Omega_{Y/\mathbb{k}}^1) \times H^1(Y, \Omega_{Y/\mathbb{k}}^1) \rightarrow H^2(Y, \Omega_{Y/\mathbb{k}}^2) \simeq \mathbb{k},$$

the value $\langle c(\mathcal{L}_1), c(\mathcal{L}_2) \rangle$ in \mathbb{k} equals the intersection number $\mathcal{L}_1 \cdot \mathcal{L}_2$ modulo $\text{char}(\mathbb{k})$ for any invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ on Y (cf. [18, Chapter. V, Exer. 1.8]). Now, $\text{Pic}(Y)$ is a unimodular lattice for the intersection pairing, since Y is rational. As a consequence, c induces an injection $\text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{k} \rightarrow H^1(Y, \Omega_{Y/\mathbb{k}}^1)$. Hence, g_1, \dots, g_k , and f_2 are linearly independent by (C4). Therefore, the connecting homomorphism for (VI-1) is injective, and we have

$$H^0(Y, \Omega_{Y/\mathbb{k}}^1(\log(\bar{G} + \bar{F}_2))) = H^0(Y, \Omega_{Y/\mathbb{k}}^1) = 0. \quad \square$$

Step 3. We consider the blowing up $\tau: Z \rightarrow Y$ at the two nodes P_1 and P_2 of \bar{F}_1 and \bar{F}_2 , respectively. Let F_i be the proper transform of \bar{F}_i in Z and let J_i be the exceptional divisor $\tau^{-1}(P_i)$ for $i = 1, 2$. Then, $F_i \rightarrow \bar{F}_i$ is the normalization map, and $\tau^*(\bar{F}_i) = F_i + 2J_i$ for $i = 1, 2$. In particular,

$$(VI-2) \quad K_Z \sim -F_1 - J_1 + J_2, \quad \text{and} \quad -2K_Z \sim F_1 + F_2.$$

Let G_i and S_j denote the proper transforms of \bar{G}_i and \bar{S}_j in Z for $1 \leq i \leq k$ and $1 \leq j \leq l$. Here, G_i is the total transform of \bar{G}_i , since $\bar{G}_i \cap \bar{F} = \emptyset$. We set $F := F_1 + F_2$, $G := \sum_{i=1}^k G_i$, and $S := \sum_{j=1}^l S_j$. We require the following conditions:

(C5) $S + F + J_1 + J_2$ is a simple normal crossing divisor.

(C6) S_j are (-1) -curves for any $1 \leq j \leq l$.

Let B^+ be the total transform of \bar{B}^+ in Z . Then, B^+ is also a simple normal crossing divisor by (C3) and (C5). We define B to be a reduced divisor such that $S + G + F \leq B \leq S + G + F + J_1 + J_2 = B^+$.

Lemma 6.5. *In the situation of Step 3,*

$$H^2(Z, \Theta_{Z/\mathbb{k}}(-\log(G + F))) = H^2(Z, \Theta_{Z/\mathbb{k}}(-\log B)) = 0.$$

Proof. The second vanishing is derived from the first, since we have an exact sequence

$$0 \rightarrow \Theta_{Z/\mathbb{k}}(-\log B) \rightarrow \Theta_{Z/\mathbb{k}}(-\log(G + F)) \rightarrow \bigoplus_{j=1}^l \mathcal{O}_{S_j}(S_j) \oplus \bigoplus_{J_i \subset B} \mathcal{O}_{J_i}(J_i) \rightarrow 0$$

for the simple normal crossing divisors B and $G + F$, in which $H^1(\mathcal{O}_{S_j}(S_j)) = H^1(\mathcal{O}_{J_i}(J_i)) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ by (C6). The first vanishing is equivalent to

$$H^0(Z, \Omega_{Z/\mathbb{k}}^1(\log(G + F)) \otimes \mathcal{O}_Z(K_Z)) = 0$$

by Serre duality. By (VI-2), we have an inclusion

$$\begin{aligned} \tau_* \left(\Omega_{Z/\mathbb{K}}^1(\log(G + F)) \otimes \mathcal{O}_Z(K_Z) \right) &\hookrightarrow \tau_* \left(\Omega_{Z/\mathbb{K}}^1(\log(G + F_2)) \otimes \mathcal{O}_Z(J_2) \right) \\ &\hookrightarrow \Omega_{Y/\mathbb{K}}^1(\log(\bar{G} + \bar{F}_2)), \end{aligned}$$

where the last map is the injection to the double-dual. Thus, we are done by Lemma 6.4. \square

Step 4. We take a successive blowings up $\varphi: M \rightarrow Z$ whose centers are certain nodes of the total transform of B^+ . On the choice of nodes, we assume that the total transform $B_M = \varphi^{-1}(B)$ of B in M contains a disjoint union $D = \bigcup_{i=1}^m D^{(i)}$ of linear chains $D^{(1)}, \dots, D^{(m)}$ of smooth rational curves satisfying the following conditions (C7)–(C11):

(C7) $\varphi(D) = B$.

(C8) Each $D^{(i)}$ contains no (-1) -curve, and is contractible to a toric singularity of class T.

Let $\mu: M \rightarrow X$ be the contraction morphism of D . Then, X has only toric singularities of class T and μ is the minimal resolution of singularities. Let Δ be the \mathbb{Q} -divisor supported on D define by $K_M + \Delta = \mu^*(K_X)$.

(C9) $K_X^2 = K^2$.

(C10) $\Delta\Gamma > 1$ for any φ -exceptional (-1) -curve Γ on M .

(C11) For any $D^{(i)}$, there exist smooth rational curves Γ and Γ' not contained in $D^{(i)}$ such that

- $\Gamma + D^{(i)} + \Gamma'$ is a linear chain of smooth rational curves with the end components Γ and Γ' ,
- $\varphi(\Gamma \cup \Gamma')$ is contained in B^+ .

We do not require the following condition (A3) which is however useful for checking the ampleness of K_X (cf. Proposition 6.6.(4) below):

(A3) There is no φ -exceptional (-2) -curve contained in $M \setminus D$.

Proposition 6.6. *In the situation of Step 4, the following hold:*

- (1) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.
- (2) $H^2(M, \Theta_{M/\mathbb{K}}(-\log D)) = H^2(X, \Theta_X) = 0$.
- (3) K_X is nef and big, and it is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor whose support is the union of D and the φ -exceptional locus.
- (4) K_X is ample if and only if there is no (-2) -curve contained in $M \setminus D$. If (A1) and (A3) are satisfied, then K_X is ample. If K_X is ample, then (A2) and (A3) are satisfied. In case B contains J_1 or J_2 , K_X is ample if and only if (A2) and (A3) are satisfied.

Proof. (1) follows from that X has only rational singularities and M is rational.

(2): By Corollary 2.12, it suffices to prove $H^2(M, \Theta_{M/\mathbb{k}}(-\log D)) = 0$. Now $B_M = \varphi^{-1}(B)$ is a simple normal crossing divisor containing D . Since the cokernel of the natural injection $\Theta_{M/\mathbb{k}}(-\log B_M) \hookrightarrow \Theta_{M/\mathbb{k}}(-\log D)$ is supported on the one-dimensional subscheme $B_M - D$, it is enough to prove: $H^2(M, \Theta_{M/\mathbb{k}}(-\log B_M)) = 0$. We have an isomorphism $\Theta_{M/\mathbb{k}}(-\log B_M) \simeq \varphi^* \Theta_{Z/\mathbb{k}}(-\log B)$, since the center of the each step of the successive blowings up φ is a node of the total transform of B . Hence, by Lemma 6.5, we have

$$H^2(M, \Theta_{M/\mathbb{k}}(-\log B_M)) \simeq H^2(Z, \Theta_{Z/\mathbb{k}}(-\log B)) = 0.$$

(3): By (VI-2), $K_Z + (1/2)(F_1 + F_2) \sim_{\mathbb{Q}} 0$. Since $F_1 + F_2$ is smooth, the pair $(Z, (1/2)(F_1 + F_2))$ is terminal (in the sense of [27, Definition 1.16]). Hence, $K_M + (1/2)(F'_1 + F'_2) \sim_{\mathbb{Q}} A$ for an effective \mathbb{Q} -divisor A whose support is just the exceptional locus for $\varphi: M \rightarrow Z$. Thus,

$$(VI-3) \quad K_M + \Delta \sim_{\mathbb{Q}} A + \Delta - (1/2)(F'_1 + F'_2).$$

Here, the right hand side is an effective \mathbb{Q} -divisor whose support is the union of D and the φ -exceptional locus, since

$$\text{mult}_{F'_i}(\Delta) > 1/2 \quad \text{for } i = 1, 2,$$

by $F_i'^2 \leq F_i^2 = -4$ and by Corollary 3.7.2 and Lemma 3.6. If $K_M + \Delta$ is nef, then this is also big by (C9). Thus, it suffices to derive a contradiction assuming that $K_M + \Delta$ is not nef. Then, there is a curve Γ with $(K_M + \Delta)\Gamma < 0$. By (VI-3), Γ is φ -exceptional or μ -exceptional. But Γ is not μ -exceptional; for otherwise, $(K_M + \Delta)\Gamma = \mu^*(K_X)\Gamma = 0$. Thus, Γ is φ -exceptional. Hence, $\Delta\Gamma \geq 0$ and $K_M\Gamma < 0$; consequently, Γ is a (-1) -curve with $\Delta\Gamma < 1$. This is a contradiction to (C10).

(4): By Nakai–Moishezon's criterion, K_X is not ample if and only if there is a curve Γ such that it is not μ -exceptional and $(K_M + \Delta)\Gamma = 0$. Let Γ be such a curve. Then, $\Gamma^2 < 0$ by the Hodge index theorem, and moreover, $K_M\Gamma = \Delta\Gamma = 0$ by (C10). Thus, Γ is a (-2) -curve contained in $M \setminus D$. Conversely, if Γ is a (-2) -curve in $M \setminus D$, then K_X is not ample by $K_X\varphi_*(\Gamma) = (K_M + \Delta)\Gamma = 0$.

Assume that the (-2) -curve Γ above is not φ -exceptional. Then, Γ does not intersect the φ -exceptional locus, since $A\Gamma = 0$ by (VI-3). Hence, $\varphi(\Gamma) \cap \varphi(D) = \emptyset$. In particular, $\tau\varphi(\Gamma) \subset Y \setminus (\bar{S} \cup \bar{G})$. Therefore, (A1) and (A3) imply the ampleness of K_X . If B contains J_1 , then $\varphi(\Gamma)$ does not intersect the fiber $F_1 \cup J_1$; hence, $\tau\varphi(\Gamma)$ is a (-2) -curve contained in a fiber of $Y \rightarrow \mathbb{P}^1$ and is contained in $Y \setminus (\bar{S} \cup \bar{G})$. Thus, (A2) and (A3) imply the ampleness of K_X in this case. If (A2) or (A3) fails, then we can find an irreducible curve Γ contained in $M \setminus D$; thus K_X is not ample. \square

Remark. Our proof of Proposition 6.6.(2) is different from the proof of the corresponding result in [33, Section 4], but has many common ideas.

Lemma 6.7. *In Step 4, the condition (C10) is derived from the following three conditions for all the φ -exceptional (-1) -curves Γ :*

- (1) Γ intersects at least two irreducible components of D .
- (2) If Γ intersects a connected component of D which defines a toric singularity of type $T(l, 2, 1)$ for some $l \geq 1$, then Γ intersects another component of D defining a toric singularity of type $T(d, n, a)$ with $n > 2$.
- (3) $\Delta\Gamma > 1$ if Γ intersects a (-2) -curve belonging to a sequence of (-2) -curves at an end of a connected component of D .

Proof. Let Γ be a φ -exceptional (-1) -curve. Assume that Γ intersects two irreducible components Ξ_1, Ξ_2 of D such that

- Ξ_i does not belong to any sequence of (-2) -curves at an end of a connected component of D for $i = 1, 2$,
- the linear chain containing Ξ_1 defines a singularity of type $T(d, n, a)$ with $n > 1$.

It suffices to prove $\Delta\Gamma > 1$. By Lemma 3.7.2, the multiplicity of Δ along Ξ_1 is greater than $1/2$ and the multiplicity along Ξ_2 is greater than or equal to $1/2$. Thus, we have $\Delta\Gamma > 1$. \square

Applying Theorem 5.2 to X above, we have:

Proposition 6.8. *Let Φ be a cubic pencil on $\mathbb{P}_{\mathbb{k}}^2$ satisfying (C1) and (C2), and let $Y \rightarrow \mathbb{P}^2$ be the elimination of the base locus of Φ by a succession of blowings up at points. Let \bar{F} , \bar{G} , \bar{S} , \bar{B} , \bar{B}^+ be divisors on Y satisfying conditions (C3) and (C4), and also conditions (C5) and (C6) on the blown up surface Z at the nodes of \bar{F} . Let $\varphi: M \rightarrow Z$ be a succession of blowings up at certain nodes of the total transform of \bar{B}^+ , and let $D = \sum D^{(i)}$ be a disjoint union of linear chains of smooth rational curves contained in the total transform of \bar{B} in which the conditions (C7)–(C11) are satisfied. Then, there is a minimal projective surface \mathbb{S} of general type defined over \mathbb{k} such that $p_g(\mathbb{S}) = q(\mathbb{S}) = 0$, $K_{\mathbb{S}}^2$ equals the given integer $1 \leq K^2 \leq 4$ in the Main Theorem, and $H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = 0$. If $M \setminus D$ contains no (-2) -curves, we can even assume the canonical divisor $K_{\mathbb{S}}$ to be ample.*

Step 5. Let B_M^+ be the total transform of B^+ in M . This is a simple normal crossing divisor consisting of rational curves. Since $\tau \circ \varphi: M \rightarrow Y$ is a succession of blowings up whose centers are nodes of the total transform of \bar{B} , by Lemma 6.2 and by the condition on \bar{S}_j in Step 2, we can realize M as the closed fiber of a smooth projective family $M_{\Lambda} \rightarrow \text{Spec } \Lambda$ of rational surfaces and B_M^+ as the closed fiber of a divisor $B_{M_{\Lambda}}^+$ on M_{Λ} flat

over Λ such that every irreducible component of $B_{M_\Lambda}^+$ is a \mathbb{P}^1 -bundle over Λ and that any non-empty intersection of two prime divisors of $B_{M_\Lambda}^+$ is a section over $\text{Spec } \Lambda$. Moreover, we have a relative Cartier divisor on M_Λ whose restriction to M is linearly equivalent to the pullback of an ample divisor on X . For, the exceptional divisors for the birational morphism $M_\Lambda \rightarrow Y_\Lambda \rightarrow \mathbb{P}_\Lambda^2$ and the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ generate $\text{Pic}(M_\Lambda)$. Then, by Proposition 6.9 below, we have a projective family $X_\Lambda \rightarrow \text{Spec } \Lambda$ of normal projective surfaces and a birational morphism $\mu_\Lambda: M_\Lambda \rightarrow X_\Lambda$ over $\text{Spec } \Lambda$ such that the closed fiber of X_Λ is isomorphic to X and the morphism μ_Λ restricted to the closed fibers is just the minimal resolution $\mu: M \rightarrow X$.

Proposition 6.9. *Let Λ be a discrete valuation ring with the residue field \mathbb{k} which is algebraically closed. Let V be a normal projective surface with only rational singularities defined over \mathbb{k} . Let $\widetilde{W} \rightarrow \text{Spec } \Lambda$ be a smooth projective morphism such that $W := \widetilde{W} \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{k}$ is the minimal resolution of singularities of V . Let $\{\widetilde{E}_i\}_{i \in I}$ be a set of prime divisors on \widetilde{W} such that:*

- \widetilde{E}_i is a \mathbb{P}^1 -bundle over $\text{Spec } \Lambda$ for any $i \in I$.
- $E_i = \widetilde{E}_i \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{k}$ is an irreducible component of the exceptional locus E of the minimal resolution $\nu: W \rightarrow V$.
- Conversely, any irreducible component of E is equal to E_i for a unique $i \in I$.
- $\widetilde{E}_i \cap \widetilde{E}_j$ is a section of $\widetilde{W} \rightarrow \text{Spec } \Lambda$ if $E_i \cap E_j \neq \emptyset$.

Assume that there is a divisor \widetilde{L} on \widetilde{W} such that $L := \widetilde{L}|_W$ is linearly equivalent to the pullback of an ample divisor on V . Then, there exist a normal projective Λ -scheme \widetilde{V} and a proper birational morphism $\widetilde{\nu}: \widetilde{W} \rightarrow \widetilde{V}$ satisfying the following conditions:

- (1) $V \simeq \widetilde{V} \times_{\text{Spec } \Lambda} \text{Spec } \mathbb{k}$, and ν is obtained by the base change of $\widetilde{\nu}$ by $\text{Spec } \mathbb{k} \rightarrow \text{Spec } \Lambda$.
- (2) The $\widetilde{\nu}$ -exceptional locus is $\bigcup \widetilde{E}_i$, and $\widetilde{\nu}(\widetilde{E}_i)$ is a section of $\widetilde{V} \rightarrow \text{Spec } \Lambda$ for all $i \in I$,
- (3) If rK_V is Cartier for a positive integer r , then so is $rK_{\widetilde{V}}$ and $rK_{\widetilde{V}}|_V \sim rK_V$.

Proof. We may assume that $L = \nu^*(L_V)$ for a very ample divisor L_V on V such that $H^i(V, \mathcal{O}_V(L_V)) = 0$ for all $i > 0$, by replacing L with mL for $m \gg 0$. Since V has only rational singularities, $H^i(W, \mathcal{O}_W(L)) = 0$ for any $i > 0$. The morphism defined by the base point free linear system $|L|$ is just $\nu: W \rightarrow V$ followed by a closed immersion $V \hookrightarrow \mathbb{P}^N$, where $N = \dim |L|$. We can show:

- Claim 6.9.1.*
- (i) $H^i(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(\widetilde{L})) = 0$ for any $i > 0$.
 - (ii) The natural homomorphism $H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(\widetilde{L})) \otimes_\Lambda \mathbb{k} \rightarrow H^0(W, \mathcal{O}_W(L))$ is an isomorphism.

- (iii) $\mathcal{O}_{\widetilde{W}}(\widetilde{L})$ is generated by global sections, i.e., $H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(\widetilde{L})) \otimes_{\Lambda} \mathcal{O}_{\widetilde{W}} \rightarrow \mathcal{O}_{\widetilde{W}}(\widetilde{L})$ is surjective.

Proof. In fact, (i) is a consequence of the upper semi-continuity theorem for $\widetilde{W} \rightarrow \operatorname{Spec} \Lambda$ and the vanishing $H^i(W, \mathcal{O}_W(L)) = 0$ for any $i > 0$. Then, (ii) is obtained as the base change isomorphism. The homomorphism in (iii) is surjective after tensoring \mathbb{k} over Λ , by the freeness of $|L|$ and by (ii). Thus, the homomorphism is surjective along the closed fiber of $\widetilde{W} \rightarrow \operatorname{Spec} \Lambda$, and is surjective everywhere on \widetilde{W} , since Λ is a local ring. \square

We continue the proof of Proposition 6.9. Now, by Claim 6.9.1.(ii) above, we see that $H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(\widetilde{L}))$ is a free Λ -module of rank N , and the surjection in Claim 6.9.1.(iii) defines a morphism $\widetilde{W} \rightarrow \mathbb{P}_{\Lambda}^N$. Let $\widetilde{W} \rightarrow \widetilde{V} \rightarrow \mathbb{P}_{\Lambda}^N$ be the Stein factorization of $\widetilde{W} \rightarrow \mathbb{P}_{\Lambda}^N$. Then, \widetilde{V} is normal and the fiber over $\operatorname{Spec} \mathbb{k}$ of the proper morphism $\tilde{\nu}: \widetilde{W} \rightarrow \widetilde{V}$ is just $\nu: W \rightarrow V$.

By construction, $\tilde{\nu}$ is an isomorphism on $W \setminus \bigcup E_i$; hence $\tilde{\nu}$ is a birational morphism. Every \widetilde{E}_i for $i \in I$ is $\tilde{\nu}$ -exceptional, since $\mathcal{O}_{\widetilde{W}}(\widetilde{L})|_{\widetilde{E}_i} \simeq \mathcal{O}_{\widetilde{E}_i}$. We shall show that there is no other $\tilde{\nu}$ -exceptional divisor. Let $\widetilde{\Gamma}$ be a $\tilde{\nu}$ -exceptional prime divisor on \widetilde{W} . If $\widetilde{\Gamma}$ is not flat over Λ , then $\widetilde{\Gamma}$ is contained in the closed fiber W and also contained in some E_i ; this is a contradiction, since $\operatorname{codim}(E_i, \widetilde{W}) = 2$. Thus, $\widetilde{\Gamma}$ is flat over Λ . Then, the closed fiber Γ of $\widetilde{\Gamma} \rightarrow \operatorname{Spec} \Lambda$ is a union of ν -exceptional curves. Here, the intersection number $b := \Gamma E_i$ is negative for some $i \in I$. Hence, $\widetilde{\Gamma} = \widetilde{E}_i$, since $\mathcal{O}_{\widetilde{W}}(\widetilde{\Gamma})|_{\widetilde{E}_i}$ is isomorphic to $\mathcal{O}_{\mathbb{P}_{\Lambda}^1}(b)$. Therefore, the $\tilde{\nu}$ -exceptional locus is $\bigcup \widetilde{E}_i$. Since \widetilde{E}_i is a \mathbb{P}^1 -bundle over Λ , the image $\tilde{\nu}(\widetilde{E}_i)$ is a section of $\widetilde{V} \rightarrow \operatorname{Spec} \Lambda$. Thus, all the assertions except (3) are satisfied.

For (3), it suffices to show that $rK_{\widetilde{V}}$ is Cartier. In fact, if $rK_{\widetilde{V}}$ is Cartier, then $rK_{\widetilde{V}}|_V \sim rK_V$ by Remark 3.9.3. We consider the rational numbers $b_i \geq 0$ such that $\nu^*(K_X) = K_W + \sum b_i E_i$. Then, $rb_i \in \mathbb{Z}$ for any i . Let \widetilde{D} be the Cartier divisor $rK_{\widetilde{W}} + \sum rb_i \widetilde{E}_i$ on \widetilde{W} . Then, $\widetilde{D}|_W \sim \nu^*(rK_V)$. We apply the argument above to $\widetilde{L}' = a\widetilde{L} + \widetilde{D}$ for $a \gg 0$ instead of \widetilde{L} , where $aL_V + rK_V$ is ample on V . Then, $\mathcal{O}_{\widetilde{W}}(\widetilde{L}')$ is generated by global sections and defines a morphism $\widetilde{W} \rightarrow \mathbb{P}_{\Lambda}^{N'}$ for some $N' > 0$. Let $\tilde{\nu}': \widetilde{W} \rightarrow \widetilde{V}'$ be the birational morphism obtained as the Stein factorization. Then, the exceptional locus of $\widetilde{W} \rightarrow \widetilde{V}'$ is just the union of $\bigcup \widetilde{E}_i$. By Zariski's main theorem, we have an isomorphism $\widetilde{V} \simeq \widetilde{V}'$ compatible with $\tilde{\nu}$ and $\tilde{\nu}'$. Consequently, \widetilde{D} is the pullback of a Cartier divisor on \widetilde{V} . Hence $rK_{\widetilde{V}}$ is Cartier and $\widetilde{D} = \tilde{\nu}^*(rK_{\widetilde{V}})$ by $\tilde{\nu}_*(\widetilde{D}) = \tilde{\nu}_*(rK_{\widetilde{W}}) = rK_{\widetilde{V}}$. Thus, we have finished the proof of Proposition 6.9. \square

Step 6. We can apply Theorem 5.4 to $X_{\Lambda} \rightarrow \operatorname{Spec} \Lambda$ constructed in *Step 5*. In fact, the assumptions (i) and (ii) of Theorem 5.4 are confirmed by Proposition 6.6 and by an argument in *Step 4* and the condition (C11), respectively. Let $X_{\mathbb{K}}$ be the geometric

generic fiber of $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$ for an algebraically closed field \mathbb{K} containing Λ . Thus, by Theorem 5.4 and by Corollary 5.5, we have deformations $\mathcal{X} \rightarrow T$ of X and $\mathcal{X}_{\mathbb{K}} \rightarrow T_{\mathbb{K}}$ of \overline{X} such that:

- T is a non-singular algebraic curve over \mathbb{k} , and $\mathcal{X} \rightarrow T$ satisfies the conditions (1) and (2) of Theorem 5.2.
- $T_{\mathbb{K}}$ is a non-singular algebraic curve over \mathbb{K} , and $\mathcal{X}_{\mathbb{K}} \rightarrow T_{\mathbb{K}}$ satisfies the conditions (1) and (2) of Theorem 5.2.
- $\pi_1^{\operatorname{alg}}(X_{\mathbb{K},t'}) \rightarrow \pi_1^{\operatorname{alg}}(X_t)$ is surjective for any closed smooth fibers $X_t = \mathcal{X} \times_T t$ and $X_{\mathbb{K},t'} = \mathcal{X}_{\mathbb{K}} \times_{T_{\mathbb{K}}} t'$.

In order to get the algebraic simply connectedness of $X_{\mathbb{K},t'}$, we shall compare $X_{\mathbb{K}}$ and a normal projective surface \overline{X} defined over \mathbb{C} which is constructed by the same method as in Steps 1–4.

Let $Y_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be the elliptic fibration obtained from the pencil Φ on $\mathbb{P}_{\mathbb{C}}^2$ by replacing \mathbb{k} with \mathbb{C} . Then, by the same process as in Steps 3 and 4, we have birational morphisms $\tau_{\mathbb{C}}: Z_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$, $\varphi_{\mathbb{C}}: M_{\mathbb{C}} \rightarrow Z_{\mathbb{C}}$, and a disjoint union $D_{\mathbb{C}}$ of linear chains of smooth rational curves such that the contraction of $D_{\mathbb{C}}$ is the minimal resolution $\mu_{\mathbb{C}}: M_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ of a normal projective surface $X_{\mathbb{C}}$ with only toric singularities of class T. Let \mathbb{K}_0 be a subfield of \mathbb{K} finitely generated over \mathbb{Q} such that $X_{\mathbb{K}}$, $\operatorname{Sing} X_{\mathbb{K}}$, and $\overline{X} \rightarrow \overline{T}$ are defined over \mathbb{K}_0 . We take an injection $\mathbb{K}_0 \hookrightarrow \mathbb{C}$ and set $\overline{X} := X_{\mathbb{K}_0} \times_{\operatorname{Spec} \mathbb{K}_0} \operatorname{Spec} \mathbb{C}$. Then, $\overline{X} \simeq X_{\mathbb{C}}$ by our construction of $X_\Lambda \rightarrow \operatorname{Spec} \Lambda$.

Lemma 6.10. If $X_{\mathbb{C}} \setminus \operatorname{Sing} X_{\mathbb{C}} \simeq M_{\mathbb{C}} \setminus D_{\mathbb{C}}$ is simply connected (with respect to the Euclidean topology), then a general smooth closed fiber X_t of $\mathcal{X} \rightarrow T$ is algebraically simply connected.

Proof. By the assumption, any \mathbb{Q} -Gorenstein smoothing of $X_{\mathbb{C}}$ is simply connected by an argument in the proof of [33, Theorem 3.1] using results on rational blowdown 4-manifolds, Milnor fibers, and van-Kampen's theorem. Thus, by Remark 5.5.1, a smooth closed fiber $X_{\mathbb{K},t'}$ of $\mathcal{X}_{\mathbb{K}} \rightarrow T_{\mathbb{K}}$ is algebraically simply connected. Therefore, so is X_t by Corollary 5.5. \square

The following is useful for proving the simply connectedness of $M_{\mathbb{C}} \setminus D_{\mathbb{C}}$ and used in the proof of [33, Theorem 3.1].

Lemma 6.10.1. In the construction of M and D above, assume that $\mathbb{k} = \mathbb{C}$. Then, $M \setminus D$ is simply connected (with respect to the euclidean topology) provided that, for any connected component D_i of D , there exists a smooth rational curve E on M and another connected component D_j of D such that:

- (1) E intersects an end component of D_i and an end component of D_j .

- (2) $ED_i = ED_j = 1$, and E does not intersect the other connected components of D .
- (3) $\gcd(d_i n_i, d_j n_j) = 1$ for the types $T(d_i, n_i, a_i)$ and $T(d_j, n_j, a_j)$ of the toric singularities defined by D_i and D_j , respectively.

Proof. Let Q_i be the toric singular point $\mu(D_i)$ of class T and let $T(d_i, n_i, a_i)$ be the type. We have an open neighborhood $\overline{\mathcal{U}}_i$ (with respect to the euclidean topology) of Q_i and a finite surjective morphism $\lambda_i: \mathcal{U}_i^\sim \rightarrow \overline{\mathcal{U}}_i$ such that

- $\overline{\mathcal{U}}_i \cap \text{Sing } X = \{Q_i\}$,
- $\overline{\mathcal{U}}_i$ is topologically contractible,
- \mathcal{U}_i^\sim is a non-singular surface topologically contractible to a point,
- λ_i is étale over $\overline{\mathcal{U}}_i \setminus \{Q_i\}$,
- λ_i is a cyclic covering of degree $d_i n_i^2$.

In particular, $\pi_1(\overline{\mathcal{U}}_i \setminus \{Q_i\})$ is a cyclic group of order $d_i n_i^2$. We set $\mathcal{U}_i := \mu^{-1}(\overline{\mathcal{U}}_i)$, which is homotopic to D_i . Hence, \mathcal{U}_i is simply connected. Moreover, $\mathcal{U}_i \setminus D = \mathcal{U}_i \setminus D_i \simeq \overline{\mathcal{U}}_i \setminus Q_i$. We may assume that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for any i, j . For the open immersion $\mathcal{U}_i \setminus D \hookrightarrow M \setminus D$, we have a homomorphism $\pi_1(\mathcal{U}_i \setminus D) \rightarrow \pi_1(M \setminus D)$ defined up to conjugate by considering a path connecting the reference points of $\mathcal{U}_i \setminus D$ and $M \setminus D$. Let γ_i be the image of a generator of $\pi_1(\mathcal{U}_i \setminus D)$ in $\pi_1(M \setminus D)$. Then, $\pi_1(M \setminus D)$ is generated by the images γ_i for any i by van-Kampen's theorem applied to the open covering $M = (M \setminus D) \cup \bigcup \mathcal{U}_i$, since M and \mathcal{U}_i are simply connected.

For a connected component D_i , let D_j be another connected component and E be a smooth rational curve satisfying (1)–(3). Then, $E \setminus D \simeq \mathbb{C} \setminus \{0\}$. Let γ_E be the image of a generator of the cyclic group $\pi_1(E \setminus D)$ in $\pi_1(M \setminus D)$ by a homomorphism $\pi_1(E \setminus D) \rightarrow \pi_1(M \setminus D)$ defined up to conjugate similarly to the above from the inclusion $E \setminus D \hookrightarrow M \setminus D$. Then, γ_E is conjugate to γ_i or γ_i^{-1} , and conjugate to γ_j or γ_j^{-1} , since E intersects D_i (resp. D_j) transversely only at one point which is in an end component. Therefore, γ_i is conjugate to γ_j or γ_j^{-1} . Hence, $\gamma_i = \gamma_j = 1$ by (3). This proves that $M \setminus D$ is simply connected. \square

By the discussion in Steps 5 and 6, we have:

Proposition 6.11. *In the situation of Proposition 6.8, let $M_{\mathbb{C}}$ be the same surface obtained as above by replacing \mathbb{k} with \mathbb{C} , and let $D_{\mathbb{C}}$ be the same linear chain as D on $M_{\mathbb{C}}$. If $M_{\mathbb{C}} \setminus D_{\mathbb{C}}$ is simply connected in addition, then one can require the surface \mathbb{S} in Proposition 6.8 to be algebraically simply connected.*

7. PROOF OF MAIN THEOREM

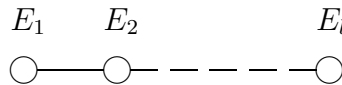
We shall prove the Main Theorem by giving explicit examples using the method in Section 6. In Examples 7.1–7.8 below, the necessary tasks are:

- Giving assumptions on $\text{char}(\mathbb{k})$ and K^2 .
- Defining two cubic homogeneous polynomials ϕ_0 and ϕ_∞ in $\mathbb{Z}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ and checking the conditions (C1) and (C2) (cf. *Step 1* in Section 6).
- Defining the divisors \bar{F} , \bar{S} , and \bar{G} on Y , and checking (C3) and (C4). For the ampleness of K_{X_t} , we check (A1) or (A2) (cf. *Step 2* in Section 6).
- Defining the divisor B on Z after checking (C5) and (C6) (cf. *Step 3* in Section 6).
- Defining the birational morphism $\varphi: M \rightarrow Y$ and the union D of linear chains of rational curves on M satisfying (C7)–(C11). For the ampleness of K_{X_t} , we check (A3) (cf. *Step 4* in Section 6).
- If possible, proving the ampleness of K_X (equivalent to the absence of (-2) -curves on $M \setminus D$) using (A1)–(A3), or another argument (cf. Proposition 6.6.(4)).
- Proving that $M \setminus D$ is simply connected (with respect to the euclidean topology) when $\mathbb{k} = \mathbb{C}$ using Lemma 6.10.1 or referring to papers [33], [42] and [43].

Having done these tasks, we obtain the desired surfaces by Propositions 6.8 and 6.11. The proof of the Main Theorem is written at the end.

Notation. • Let $(\mathbf{x} : \mathbf{y} : \mathbf{z})$ be a homogeneous coordinate of \mathbb{P}^2 .

- A $(-k)$ -curve means a non-singular rational curve with self-intersection number $-k$.
- The symbol $\text{LC}(b_1, \dots, b_l)$ expresses a linear chain $E = \sum_{i=1}^l E_i$ of smooth rational curves with the dual graph



such that $E_i^2 = -b_i$ for $1 \leq i \leq l$. Here, E_i is called the i -th component.

- For a non-zero regular section ϕ of an invertible sheaf, $(\phi)_0$ denotes the divisor of zeros of ϕ .
- The configuration type of singular fibers for a minimal elliptic fibration is the list of singular fibers written by Kodaira's symbol (cf. [25, Theorem 6.2]).

Example 7.1. Assume that $\text{char}(\mathbb{k}) \neq 2, 3$, and we set $K^2 = 2$. Here, we use essentially the same construction as in [33, Section 3]. We set

$$\phi_0 := (3\mathbf{x}\mathbf{y} - \mathbf{z}^2)\mathbf{z} \quad \text{and} \quad \phi_\infty := (\mathbf{x} + \mathbf{y})^3.$$

Then, $\Phi_0 = (\phi_0)_0 = Q + L_1$ and $\Phi_\infty = (\phi_\infty)_0 = 3L_2$ for the conic $Q := (3xy - z^2)_0$ and for the lines $L_1 := (z)_0$ and $L_2 := (x + y)_0$. Thus, (C1) holds. Moreover, (C2) also holds; In fact, for $c \neq 0$, the divisor $\Phi_c = (\phi_0 + c\phi_\infty)_0$ is singular if and only if $c = \pm 1/2$, where $\Phi_{1/2}$ and $\Phi_{-1/2}$ are nodal rational curves with the nodes at $(1 : -1 : 1)$ and $(1 : 1 : 1)$, respectively. On the minimal elliptic fibration $\pi: Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ defined by Φ , the configuration type of singular fibers is (IV^*, I_2, I_1, I_1) , and the (-1) -curves exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ are mutually disjoint sections \bar{S}_1, \bar{S}_2 , and \bar{S}_3 as in Figure 1. Let \bar{F}_1 and \bar{F}_2 be

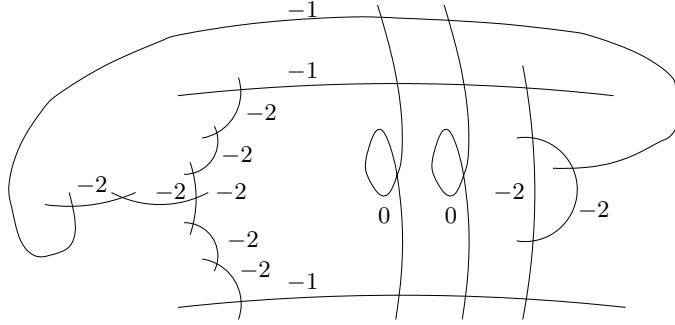


FIGURE 1. The rational elliptic surface Y in Example 7.1

the proper transforms of $\Phi_{1/2}$ and $\Phi_{-1/2}$ in Y , respectively. These are the singular fibers of type I_1 , and we define $\bar{F} := \bar{F}_1 + \bar{F}_2$. The singular fiber of type I_2 is the union of the proper transforms Q^\sim and L_1^\sim of Q and L_1 in Y , respectively. The central irreducible component of the singular fiber of type IV^* , which meets three other components, is just the proper transform L_2^\sim of L_2 in Y . We may assume that \bar{S}_1, \bar{S}_2 , and \bar{S}_3 are contracted to the points $(1 : -1 : \sqrt{-3})$, $(1 : -1 : -\sqrt{-3})$, and $(1 : -1 : 0)$, respectively, where $Q \cap L_2 = \{(1 : -1 : \sqrt{-3}), (1 : -1 : -\sqrt{-3})\}$, and $L_1 \cap L_2 = (1 : -1 : 0)$. Hence, \bar{S}_1 and \bar{S}_2 intersect Q^\sim , and \bar{S}_3 intersects L_1^\sim . We define \bar{S} to be $\bar{S}_1 + \bar{S}_2 + \bar{S}_3$. The union \bar{G}^+ of all the (-2) -curves on Y is just the union of the singular fibers of type IV^* and I_2 . Hence, $\bar{B}^+ = \bar{F} + \bar{G}^+ + \bar{S}$ satisfies the condition (C3). We define \bar{G} to be the union of the (-2) -curves except for two (-2) -curves: One is L_1^\sim and the other is the irreducible component of the singular fiber of type IV^* next to the end component meeting \bar{S}_1 . Then, \bar{G} satisfies (C4). This is shown as follows: Now, \bar{G} has three connected components whose dual graphs are of type A_5, A_1 , and A_1 . Hence, $\det(\bar{G}_i \bar{G}_j) = -24 \not\equiv 0 \pmod{\text{char}(\mathbb{k})}$ by $\det A_n = (-1)^n(n+1)$. Thus, we have (C4) by Lemma 6.3.

We shall prove (A1). Let $V \rightarrow \mathbb{P}_{\mathbb{k}}^2$ be the blowing up at the point $(1 : -1 : \sqrt{-3}) \in Q \cap L_2$. Then, V is the Hirzebruch surface of degree one and the proper transform Q' of Q in V is ample. For the total transform L'_1 of L_1 in V , we have an isomorphism from $Y \setminus (\bar{S} \cup \bar{G})$ to the affine open subset $V \setminus (Q' \cup L'_1)$. Thus, (A1) holds.

By Figure 1, we see that (C5) is true with the exceptional divisors J_1 and J_2 over the nodes, and that (C6) is also true. We define $B := S + F + G$, i.e., B does not contain J_1 and J_2 .

We take a birational morphism $\varphi: M \rightarrow Z$ so that the total transform B_M^+ of \bar{B}^+ in M as in Figure 2. Note that φ is determined uniquely by this figure. A detailed construction

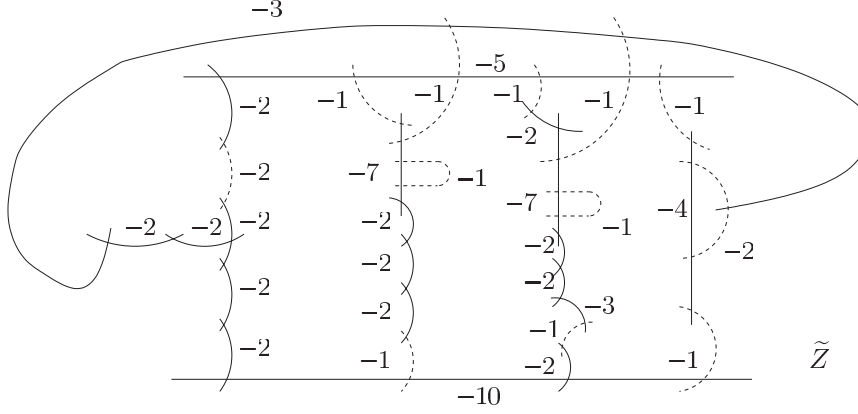


FIGURE 2. The rational surface M in Example 7.1

of φ is written in [33, Section 3]. In particular, $\rho(M) = \rho(Z) + 16 = 28$ and $K_M^2 = -18$. Here, we can find a disjoint union D of the following five linear chains of smooth rational curves in $\varphi^{-1}(B)$:

$$\begin{aligned} D_1 &= \text{LC}(2, 10, 2, 2, 2, 2, 3), & D_2 &= \text{LC}(2, 7, 2, 2, 3), \\ D_3 &= \text{LC}(7, 2, 2, 2), & D_4 &= \text{LC}(5, 2), & D_5 &= \text{LC}(4) \end{aligned}$$

Then, (C7) is checked by comparing Figures 1 and 2. Moreover, (C8) is also true: In fact, D_1, D_2, \dots, D_5 define toric singularities of types $T(1, 15, 8), T(1, 9, 5), T(1, 5, 1), T(1, 3, 1), T(1, 2, 1)$, respectively (cf. Table 1). Moreover, using Table 1, we can check (C9) by:

$$\begin{aligned} K_X^2 &= K_M^2 + \delta(1, 15, 8) + \delta(1, 9, 5) + \delta(1, 5, 1) + \delta(1, 3, 1) + \delta(1, 2, 1) \\ &= -18 + 8 + 5 + 4 + 2 + 1 = 2. \end{aligned}$$

The condition (C11) is also checked by Figure 2. We shall show (C10). By Lemma 6.7 and by Figure 2, it suffices to check $\Delta E_i > 1$ for the (-1) -curves E_1, E_2 , and E_3 characterized by:

- E_1 joins the (-10) -curve of D_1 and the end (-2) -component of D_3 .
- E_2 joins the end (-2) -component of D_2 and (-5) -curve of D_4 .
- E_3 joins the end (-2) -component of D_1 and the (-3) -curve of D_2 .

We can calculate ΔE_i for $i = 1, 2, 3$ using Table 1, where the multiplicity c_i of the i -th irreducible component of the linear chain of type $T(d, n, a)$ equals $1 - r_i/n$. Thus,

$$\begin{aligned}\Delta E_1 &= (1 - 1/15) + (1 - 4/5) > 1, & \Delta E_2 &= (1 - 5/9) + (1 - 1/3) > 1, \text{ and} \\ \Delta E_3 &= (1 - 8/15) + (1 - 4/9) > 1.\end{aligned}$$

Hence, (C10) is satisfied. Looking at Figure 2, we have (A3) immediately. Thus, K_X is ample by (A1), (A3), and by Proposition 6.6.(4). The simply connectedness of $X \setminus \text{Sing } X \simeq M \setminus D$ has been shown in the proof of [33, Theorem 3.1]. Thus, we have done all the tasks.

Example 7.2. Assume that $\text{char}(\mathbb{k}) \neq 2$ and we set $K^2 = 2$. We follow the construction in [33, Section 6, Construction]. We set

$$\phi_0 := y^2z - x^2(x - z) \quad \text{and} \quad \phi_\infty := \begin{cases} (x + z)z^2, & \text{if } \text{char}(\mathbb{k}) \neq 5, \\ (x + 2z)z^2, & \text{if } \text{char}(\mathbb{k}) = 5. \end{cases}$$

Then, $\Phi_\infty = (\phi_\infty)_0 = L_1 + 2L_2$ for the lines $L_1 = (x + z)_0$ and $L_2 = (z)_0$, and $\Phi_0 = (\phi_0)_0$ is a nodal rational cubic curve with node at $(0 : 0 : 1)$. Furthermore, L_2 is the tangent line of Φ_0 at an inflection point $(0 : 1 : 0) = L_1 \cap L_2$. In particular, (C1) holds. Moreover, (C2) also holds: In fact, we can check the following:

- For $c \neq 0$, the divisor $\Phi_c = (\phi_0 + c\phi_\infty)_0$ is singular if and only if $c = (11 \pm 5\sqrt{5})/2$ when $\text{char}(\mathbb{k}) \neq 5$ and $c = 1 \pm \sqrt{3}$ when $\text{char}(\mathbb{k}) = 5$.
- Let c_\pm be the constants $(11 \pm 5\sqrt{5})/2$ when $\text{char}(\mathbb{k}) \neq 5$ and $1 \pm \sqrt{3}$ when $\text{char}(\mathbb{k}) = 5$. Then, the node of Φ_{c_\pm} is $(-(1 \pm \sqrt{5})/2 : 0 : 1)$ when $\text{char}(\mathbb{k}) \neq 5$ and $(\pm 2\sqrt{3} : 0 : 1)$ when $\text{char}(\mathbb{k}) = 5$.

On the minimal elliptic fibration $\pi : Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ defined by Φ , the configuration type of singular fibers is $(\text{III}^*, I_1, I_1, I_1)$, and the (-1) -curves exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ are mutually disjoint sections \bar{S}_1, \bar{S}_2 , and \bar{S}_3 as in Figure 3. Let \bar{F}_1 and \bar{F}_2 be the proper transforms of Φ_{c_+} and Φ_{c_-} in Y , respectively. These are singular fibers of type I_1 . We define $\bar{F} := \bar{F}_1 + \bar{F}_2$. We may assume that \bar{S}_1, \bar{S}_2 , and \bar{S}_3 are contracted to the points $(0 : 1 : 0)$, $(1 : \sqrt{-2} : -1)$, and $(1 : -\sqrt{-2} : -1)$, respectively, where $\Phi_0 \cap L_2 = L_1 \cap L_2 = \{(0 : 1 : 0)\}$ and $\Phi_0 \cap L_1 = \{(0 : 1 : 0), (1 : \pm\sqrt{-2} : -1)\}$. An end component of the singular fiber of type III^* is contracted to $(0 : 1 : 0)$ and meets \bar{S}_1 . Another end component is the proper transform L_1^\sim of L_1 in Y and meets \bar{S}_2 and \bar{S}_3 . The other end component is the proper transform L_2^\sim of L_2 in Y . We define $\bar{S} := \bar{S}_1 + \bar{S}_2 + \bar{S}_3$. The union \bar{G}^+ of (-2) -curves on Y is just the support of the singular fiber of type III^* . Thus, (C3) holds for $\bar{B}^+ = \bar{F} + \bar{G}^+ + \bar{S}$. We define $\bar{G} := \bar{G}^+ - L_2^\sim$. Then, \bar{G} is connected with the dual graph

holds. We have (C9) from the calculation

$$\begin{aligned} K_X^2 &= K_M^2 + \delta(1, 25, 17) + \delta(1, 19, 13) + \delta(1, 5, 1) + 2\delta(1, 4, 1) \\ &= -22 + 10 + 8 + 4 + 2 = 2 \end{aligned}$$

using Tables 1 and 2. The condition (C11) is checked by looking at Figure 4. We shall prove (C10). By Lemma 6.7, it is enough to check $\Delta E_i > 1$ for the φ -exceptional (-1) -curves E_1, E_2, E_3 characterized by:

- E_1 joins the (-11) -curve in D_1 and the end (-2) -curve in D_3 .
- E_2 joins the (-9) -curve in D_2 and the end (-2) -curve of D_2 .
- E_3 joins (-4) -curve in D_2 and the end (-2) -curve in D_1 .

Then, by Tables 1 and 2, we have:

$$\begin{aligned} \Delta E_1 &= (1 - 1/25) + (1 - 4/5) > 1, & \Delta E_2 &= (1 - 1/19) + (1 - 13/19) > 1, \\ \Delta E_3 &= (1 - 6/19) + (1 - 17/25) > 1. \end{aligned}$$

Thus, (C10) is satisfied. The condition (A3) is checked by looking at Figure 4. Thus, K_X is ample by (A1), (A3), and Proposition 6.6.(4).

We shall prove that $M \setminus D$ is simply connected when $\mathbb{k} = \mathbb{C}$ (the proof is omitted in [33, Section 6]). We apply Lemma 6.10.1. Let $T(d_i, n_i, a_i)$ be the type of the singular point defined by D_i . Then, $d_1 = \cdots = d_5 = 1$ and $(n_1, \dots, n_5) = (25, 19, 5, 2, 2)$. Looking at Figure 4, we have:

- the (-1) -curve E_3 which meets end components of D_1 and D_2 .
- a (-1) -curve meeting the end (-4) -component of D_1 and D_4 (resp. D_5).
- a (-1) -curve meeting the end (-7) -component of D_3 and D_4 .

Since $\gcd(n_1, n_2) = \gcd(25, 19) = 1$, $\gcd(n_1, n_4) = \gcd(n_1, n_5) = \gcd(25, 2) = 1$, and $\gcd(n_3, n_4) = \gcd(5, 2) = 1$, the conditions of Lemma 6.10.1 are all satisfied, and hence $M \setminus D$ is simply connected. Therefore, we have done all the tasks.

Remark 7.2.1. It is known by [22] that there exist rational elliptic surfaces whose singular fibers are of configuration type $(\text{III}^*, \text{I}_1, \text{I}_1, \text{I}_1)$ in characteristic 3.

Example 7.3. Assume that $\text{char}(\mathbb{k}) \neq 2$ and set $K^2 = 1$. We consider the same cubic pencil Φ generated by Φ_0 and Φ_∞ in Example 7.2. Thus, (C1) and (C2) hold. On the elliptic fibration $Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$, let $\bar{F} = \bar{F}_1 + \bar{F}_2$ and $\bar{S} = \bar{S}_1 + \bar{S}_2 + \bar{S}_3$ be the same as in Example 7.2, but here we define \bar{G} to be \bar{G}^+ minus two irreducible components of the singular fiber of type III^* ; one is L_2^\sim and the other is the component next to L_1^\sim . Then, (C3) and (C4) are also satisfied. For the blowing up $V \rightarrow \mathbb{P}^2$ at $(0 : 1 : 0) = L_1 \cap L_2 \cap \Phi_0$ and for the proper transform L'_1 of L_1 in V , we see that $Y \setminus (\bar{S} \cup \bar{G})$ is isomorphic to $V \setminus L'_1$

We take the birational morphism $\varphi: M \rightarrow Z$ by the following steps: (i)–(iii):

- (i) First, we blow up at the two intersection points $F_1 \cap S_2$ and $F_1 \cap S_3$.
- (ii) We blow up 4 times successively at the intersection point of F_1 and the proper transform of S_1 . Then, we have a linear chain $\text{LC}(7, 2, 2, 2)$ of rational curves.
- (iii) Applying the same process in (i) and (ii) to F_2 , we obtain a rational surface M with $\rho(M) = \rho(Z) + 12 = 24$ and $K_M^2 = -14$.

The figure consists of three side-by-side plots illustrating the spectral evolution of the operator H_{α}^{β} as α increases from 0 to 1. Each plot has a horizontal axis representing the eigenvalue and a vertical axis representing α .
 - The leftmost plot ($\alpha \in [0, 1]$) shows several horizontal lines representing constant eigenvalues. Solid lines are at -9 , -7 , and -2 . Dotted lines branch off from these solid lines at specific values of α . Labels include -9 , -7 , -2 , and -3 .
 - The middle plot ($\alpha \in [0, 1]$) focuses on the region around -7 and -2 . It shows a solid line at -7 and another at -2 . A dotted curve branches off from the -7 line. Labels include -7 , -2 , and -1 .
 - The rightmost plot ($\alpha \in [0, 1]$) focuses on the region around -2 . It shows a solid line at -2 and a dotted curve branching off from it. Labels include -2 and -1 .

Here, we see that M has a disjoint union D of the following four linear chains of smooth rational curves:

Then, (C7) and (C8) hold: In fact, D_1, \dots, D_4 define toric singularities of types $T(1, 5, 1)$, $T(1, 5, 1)$, $T(3, 2, 1)$, $T(1, 7, 1)$, respectively (cf. Table 1). We have (C9) by the calculation

using Table 1. We can check $(\mathcal{C}11)$ and $(\mathcal{A}3)$ by Figure 5. We shall prove $(\mathcal{C}10)$. By Lemma 6.7, it is enough to show $\Delta E_i > 1$ for $i = 1, 2$ for the (-1) -curves E_1, E_2 such that E_i joins the (-9) -curve of D_4 and the end (-2) -curve of D_i for $i = 1, 2$. By using Table 1, we have

$$\Delta E_1 = \Delta E_2 = (1 - 1/7) + (1 - 4/5) > 1.$$

Thus, (C10) holds. The ampleness of K_X is proved as follows. Assuming the contrary, we have a (-2) -curve Γ contained in $M \setminus D$ by Proposition 6.6.(4). Then, Γ is not φ -exceptional by (A3), and the projective curve $\bar{\Gamma} := \tau(\varphi(\Gamma))$ is contained in $Y \setminus (\bar{G} \cup \bar{S})$. By the property of this open subset of Y discussed above, $\bar{\Gamma}$ is a non-singular rational curve with self-intersection number zero and the image of $\bar{\Gamma}$ in $\mathbb{P}_{\mathbb{k}}^2$ is a line L passing through $(0 : 1 : 0)$. Since $\varphi(\Gamma)$ is a (-2) -curve, $\bar{\Gamma}$ is a bisection of $Y \rightarrow \mathbb{P}^1$ passing through the nodes of \bar{F}_1 and \bar{F}_2 . Hence, L is a line passing through the nodes of $\Phi_{c\pm}$; thus, $L = (y)_0$. This contradicts $L \ni (0 : 1 : 0)$. Therefore, K_X is ample.

We shall show that $M \setminus D$ is simply connected when $\mathbb{k} = \mathbb{C}$. We apply Lemma 6.10.1. Let $T(d_i, n_i, a_i)$ be the type of D_i for $1 \leq i \leq 4$. Then, $(d_1, \dots, d_4) = (1, 1, 3, 1)$ and $(n_1, \dots, n_4) = (5, 5, 2, 7)$. Looking at Figure 5, we have:

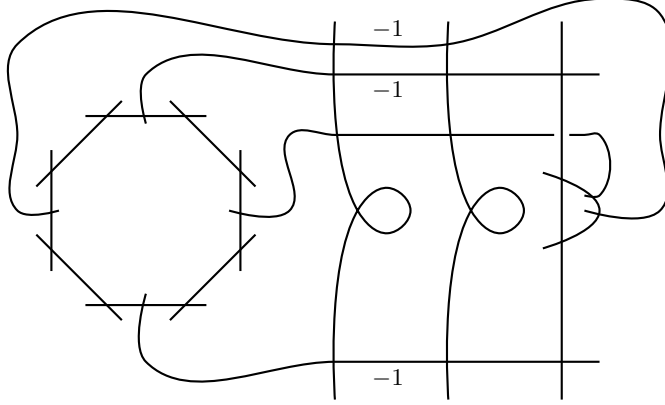
- the (-1) -curve E_4 which meets end components of D_1 and D_4 .
- the (-1) -curve E_2 which meets end components of D_2 and D_4 .
- a (-1) -curve meeting the end (-3) -component of D_3 and the end (-7) -component of D_1 .

Since $\gcd(d_1 n_1, d_4 n_4) = \gcd(d_2 n_2, d_4 n_4) = \gcd(5, 7) = 1$ and $\gcd(d_1 n_1, d_3 n_3) = \gcd(5, 6) = 1$, the conditions of Lemma 6.10.1 are all satisfied, and hence $M \setminus D$ is simply connected. Therefore, we have done all the tasks.

Example 7.4. Assume that $\text{char}(\mathbb{k}) \neq 2$, and we set $K^2 = 3$. We follow the construction in [42, Section 3]. We set

$$\phi_0 := (\mathbf{x}y + \mathbf{z}^2)(\mathbf{x} + \mathbf{y}) \quad \text{and} \quad \phi_\infty := \mathbf{x}y\mathbf{z}.$$

Then, $\Phi_0 = Q + L_4$ and $\Phi_\infty = L_1 + L_2 + L_3$ for the conic $Q = (\mathbf{x}y + \mathbf{z}^2)_0$, and the lines $L_1 = (\mathbf{x})_0$, $L_2 = (\mathbf{y})_0$, $L_3 = (\mathbf{z})_0$, and $L_4 = (\mathbf{x} + \mathbf{y})_0$. In particular, (C1) holds, and moreover, (C2) holds: In fact, for $c \neq 0$, the divisor $\Phi_c = (\phi_0 + c\phi_\infty)_0$ is singular if and only if $c = \pm 4$, where Φ_4 and Φ_{-4} are nodal rational curves with the nodes at $(1 : 1 : -1)$ and $(1 : 1 : 1)$, respectively. On the minimal elliptic fibration $\pi : Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ defined by Φ , the configuration type of singular fibers is (I_8, I_2, I_1, I_1) , and the (-1) -curves exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ are mutually disjoint four sections $\bar{S}_1, \dots, \bar{S}_4$ as in Figure 6 (cf. [42, Section 3, Figure 2] for $E(1)$ (where the symbols are different from ours)). Let \bar{F}_1 and \bar{F}_2 be the proper transforms of Φ_4 and Φ_{-4} , respectively. These are the two singular fibers of type I_1 . We define $\bar{F} := \bar{F}_1 + \bar{F}_2$. The singular fiber of type I_2 is the union of the proper transforms Q^\sim and L_4^\sim of Q and L_4 in Y . The singular fiber of type I_8 contains the proper transforms L_1^\sim , L_2^\sim , and L_3^\sim of the lines L_1 , L_2 , and L_3 in Y . The irreducible components of the singular fiber of type I_8 is labelled as $\Gamma_0 + \Gamma_1 + \dots + \Gamma_7$ as a cyclic chain, where we set $L_1^\sim = \Gamma_1$, $L_2^\sim = \Gamma_7$, and $L_3^\sim = \Gamma_4$. We may assume that $\bar{S}_1, \bar{S}_2, \bar{S}_3,$

FIGURE 6. The rational elliptic surface Y in Example 7.4

and \bar{S}_4 are contracted to the points $(0 : 0 : 1)$, $(0 : 1 : 0)$, $(1 : 0 : 0)$, and $(1 : -1 : 0)$, respectively, where $L_1 \cap L_2 \cap L_4 = \{(0 : 0 : 1)\}$, $L_1 \cap Q = L_1 \cap L_3 = \{(0 : 1 : 0)\}$, $L_2 \cap Q = L_2 \cap L_3 = \{(1 : 0 : 0)\}$, and $L_3 \cap L_4 = \{(1 : -1 : 0)\}$. Hence, \bar{S}_1 intersects L_4^\sim and Γ_0 ; \bar{S}_2 intersects Q^\sim and Γ_2 ; \bar{S}_3 intersects Q^\sim and Γ_6 ; \bar{S}_4 intersects L_4^\sim and $L_3^\sim = \Gamma_4$. We define $\bar{S} := \bar{S}_1 + \bar{S}_2 + \bar{S}_4$ removing \bar{S}_3 . Then, the condition (C3) holds. We define \bar{G} to be

$$\bar{G}^+ - \Gamma_3 - Q^\sim = \sum_{0 \leq i \leq 7, i \neq 3} \Gamma_i + L_4^\sim.$$

Then, \bar{G} has two connected components whose dual graphs are A_7 and A_1 . Thus, $\det(\bar{G}_i \bar{G}_i) = 16 \not\equiv 0 \pmod{\text{char}(\mathbb{k})}$, and hence (C4) is satisfied by Lemma 6.3. For (A1), since $\rho(Y) = 10$, it is enough to prove that \bar{S}_1 , \bar{S}_2 , and the irreducible components of \bar{G} are linearly independent in $\text{Pic}(Y)$. Assume that

$$a_1 \bar{S}_1 + a_2 \bar{S}_2 + a_3 L_4^\sim + \sum_{0 \leq i \leq 7, i \neq 3} m_i \Gamma_i \sim 0$$

for integers a_j and m_i . Then, calculating the intersection numbers with Q^\sim , L_4^\sim , \bar{S}_2 , \bar{S}_4 , and Γ_3 , we have:

$$a_2 + 2a_3 = a_1 - 2a_3 = -a_2 + m_2 = a_3 + m_4 = m_2 + m_4 = 0.$$

In particular, $a_1 = a_2 = a_3 = m_2 = m_4 = 0$. The other m_j are all zero, since $\det(\bar{G}_i \bar{G}_j) \neq 0$. Thus, we have the linear independence, and hence (A1).

We have (C5) and (C6) on Z by looking at Figure 6. We define $B := S + G + F$. We take the birational morphism $\varphi: M \rightarrow Y$ so that the total transform B_M^+ of \bar{B}^+ in M as in Figure 7 (cf. [42, Section 3, Figure 5]). In particular, $\rho(M) = \rho(Z) + 19$ and $K_M^2 = -21$. We have a disjoint union D of the following linear chains of smooth rational

Remark 7.4.1. It is known by [22] that there exist rational elliptic surfaces whose singular fibers are of configuration type (I_8, I_1, I_1, I_2) in characteristic 3. In characteristic 2, this configuration does not exist by [29].

Example 7.5. Assume that $\text{char}(\mathbb{k}) \neq 2$, and we set $K^2 = 4$. We follow the construction in [43, Section 2]. Here, we define ϕ_0 and ϕ_∞ to be the same as in Example 7.4. Thus, (C1) and (C2) hold. On the elliptic fibration $\pi: Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$, we have four sections $\bar{S}_1, \dots, \bar{S}_4$ in Example 7.4, but here, consider another horizontal curve N^\sim which is the proper transform of the line $N := (\mathbf{x} + \mathbf{y} + 2\mathbf{z})_0$ in Y . Note that N contains $(1 : -1 : 0) = L_3 \cap L_4$ and the node $(1 : 1 : -1)$ of Φ_4 . We define \bar{S} to be $\bar{S}_3 + \bar{S}_4 + N^\sim$. We also set $\bar{F} = \bar{F}_1 + \bar{F}_2$ as in Example 7.4. Then, we can check (C3) by Figure 8 (cf. [43, Figure 2]). We define

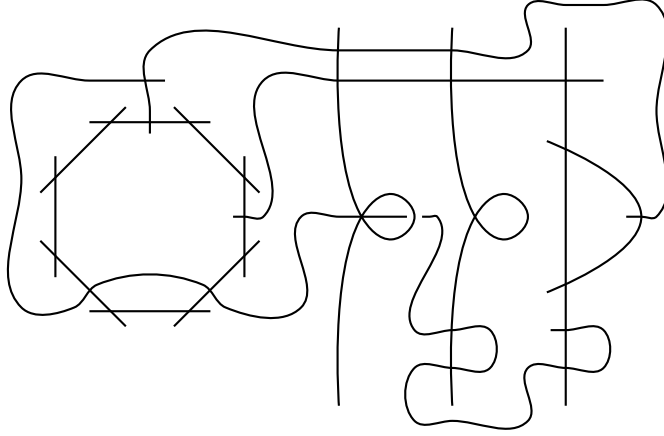


FIGURE 8. The rational elliptic surface Y in Example 7.5

the divisor \bar{G} to be

$$\bar{G}^+ - \Gamma_0 - L_2^\sim - Q^\sim = \sum_{i=1}^6 \Gamma_i + L_4^\sim.$$

Note that $\Gamma_1 = L_1^\sim$, $\Gamma_4 = L_3^\sim$, and $\Gamma_7 = L_2^\sim$. We shall prove (C4) not using Lemma 6.3. Assume that we have a linear equivalence relation

$$\sum_{i=1}^6 m_i \bar{G}_i + m_7 L_4^\sim + m_8 \bar{F}_1 \sim pH$$

for integers m_1, \dots, m_8 , $p = \text{char}(\mathbb{k})$, and a Cartier divisor H on Y . Considering the intersection numbers with Q^\sim and \bar{S}_j , we have

$$pHQ^\sim = m_7, \quad pH\bar{S}_1 = m_7 + m_8, \quad pH\bar{S}_2 = m_2 + m_8, \quad \text{and} \quad pH\bar{S}_3 = m_6 + m_8,$$

which imply $m_2 \equiv m_6 \equiv m_7 \equiv m_8 \equiv 0 \pmod{p}$. Moreover, we have $m_{i-1} + m_{i+1} \equiv 2m_i \pmod{p}$ for $2 \leq i \leq 6$ by calculating $pH\Gamma_i$. Thus, $m_i \equiv 0 \pmod{p}$ for any i , since $p \neq 2$. Hence, (C4) holds. The conditions (A2), (C5), and (C6) follow immediately from Figure 8.

We define $B := S + G + F + J_1$, where we set \bar{F}_1 to be the proper transform of Φ_4 in Y . Note that N^\sim passes through the node of \bar{F}_1 .

We take the birational morphism $\varphi: M \rightarrow Y$ by the same process as in [43, Section 2] for constructing $Z = Y \# 9\bar{\mathbb{P}}^2$ from Y , where the symbol $Y \# 9\bar{\mathbb{P}}^2$ in [43] stands for a blown up surface of Y at nine (infinitely near) points. Thus, $\rho(M) = \rho(Z) + 7 = 19$, $K_M^2 = -9$, and the total transform B_M^+ of \bar{B}^+ in M has a configuration in Figure 9 (cf. [43, Figure 3]). Here, we can find a linear chain

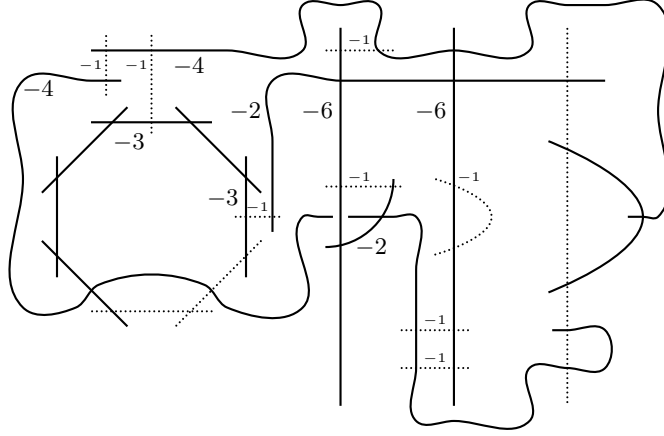


FIGURE 9. The rational surface M in Example 7.5

$$D = \text{LC}(2, 4, 6, 2, 6, 2, 4, 2, 2, 2, 3, 2, 3)$$

of smooth rational curves satisfying (C7), which corresponds to the solid lines of Figure 9. By Table 2, we see that D defines a toric singularity of type $T(1, 252, 145)$, and

$$K_M^2 = K_X^2 + \delta(1, 252, 145) = -9 + 13 = 4.$$

Thus, (C8) and (C9) hold. The conditions (C11) and (A3) follow immediately from Figure 9. We shall prove (C10) by using Lemma 6.7. Then, it suffices to show $\Delta E > 1$ for the (-1) -curve E which joins the end (-2) -component and the end (-3) -component of D . We have

$$\Delta E = (1 - 145/252) + (1 - 107/252) > 1$$

by Table 2. Thus, (C10) holds. The ampleness of K_X follows from (A2) and (A3) by Proposition 6.6.(4), since $B \supset J_1$. The simply connectedness of $M \setminus D$ has been shown in the proof of [43, Proposition 2.1]. Thus, we have done all the tasks.

Example 7.6. Assume that $\text{char}(\mathbb{k}) \neq 3$ and we set $K^2 = 1$. We define

$$\phi_0 = \mathbf{x}^2\mathbf{y} + \mathbf{y}^2\mathbf{z} + \mathbf{z}^2\mathbf{x} \quad \text{and} \quad \phi_\infty = 3\mathbf{x}\mathbf{y}\mathbf{z}.$$

Then, $\Phi_\infty = (\phi_\infty)_0 = L_1 + L_2 + L_3$ for the lines $L_1 = (\mathbf{x})_0$, $L_2 = (\mathbf{y})_0$, $L_3 = (\mathbf{z})_0$, and $\Phi_0 = (\phi_0)_0$ is a smooth cubic curve such that

$$\Phi_0|_{L_1} = P_2 + 2P_3, \quad \Phi_0|_{L_2} = 2P_1 + P_3, \quad \Phi_0|_{L_3} = P_1 + 2P_2$$

where $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$. In particular, (C1) holds, and moreover, (C2) holds: In fact, since $\text{char}(\mathbb{k}) \neq 3$, for $c \neq 0$, $\Phi_c = (\phi_0 + c\phi_\infty)_0$ is singular if and only if $c = -\omega^i$ for $i = 0, 1, 2$, where ω is a primitive cubic root of 1, and $\Phi_{-\omega^i}$ is a nodal rational curve with the node at $(1 : \omega^i : \omega^{-i})$. We also define N to be the line passing through $(0 : 1 : 0) = L_1 \cap L_3$ and the node $(1 : 1 : 1)$ of Φ_{-1} . Hence, $N = (\mathbf{x} - \mathbf{z})_0$. Thus, we have Figure 10. On the minimal elliptic fibration $\pi: Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ defined by Φ , the

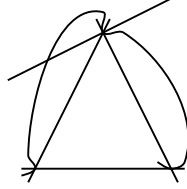


FIGURE 10. A pencil of cubics in Examples 7.6 and 7.7

configuration type of singular fibers is (I_9, I_1, I_1, I_1) , and the (-1) -curves exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ are mutually disjoint three sections \bar{S}_1 , \bar{S}_2 , and \bar{S}_3 as in Figure 11. Here, the

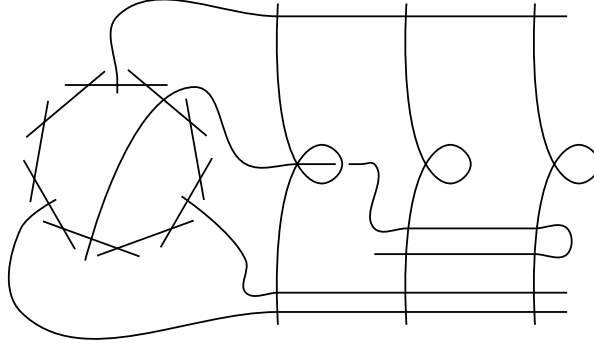


FIGURE 11. The rational elliptic surface Y in Examples 7.6 and 7.7

singular fibers of type I_1 are the proper transforms of $\Phi_{-\omega^i}$ for $i = 0, 1, 2$. We set \bar{F}_1 and \bar{F}_2 to be the proper transforms of Φ_{-1} and $\Phi_{-\omega}$, respectively, and define $\bar{F} := \bar{F}_1 + \bar{F}_2$. The irreducible components of the singular fiber of type I_9 are labeled as $\Gamma_1 + \cdots + \Gamma_9$ as a cyclic chain of smooth rational curves, where Γ_{3i} is the proper transform of L_i^\sim for $i = 1, 2, 3$. We may assume that \bar{S}_j is contracted to P_j by $Y \rightarrow \mathbb{P}_{\mathbb{k}}^2$ for $j = 1, 2, 3$. Then,

$$\bar{S}_j \Gamma_i = \begin{cases} 1, & \text{if } i \equiv 3j + 4 \pmod{9}, \\ 0, & \text{otherwise.} \end{cases}$$

The proper transform N^\sim of N in Y is a bisection of $\pi: Y \rightarrow \mathbb{P}_{\mathbb{k}}^1$ with self-intersection number zero and passing through the node of \bar{F}_1 but no nodes of other singular fibers of type I_1 . We have

$$N^\sim \bar{S}_j = N^\sim \Gamma_2 - 1 = N^\sim \Gamma_6 - 1 = N^\sim \Gamma_i = 0$$

for any $1 \leq j \leq 3$ and $1 \leq i \neq 2, 6 \leq 9$. We define $\bar{S} := \bar{S}_2 + N^\sim$. Since the union \bar{G}^+ of all the (-2) -curves on Y are the singular fiber of type I_9 , we have (C3). We define

$$\bar{G} := \bar{G}^+ - \Gamma_3 - \Gamma_4 - \Gamma_5 - \Gamma_8 - \Gamma_9 = \bar{\Gamma}_1 + \bar{\Gamma}_2 + \bar{\Gamma}_6 + \bar{\Gamma}_7.$$

Then, \bar{G} has two connected components with the dual graph A_2 . Hence, $\det(\bar{G}_i \bar{G}_j) \not\equiv 0 \pmod{\text{char}(\mathbb{k})}$, and (C4) holds by Lemma 6.3. The condition (A2) fails: In fact, $\Gamma_4 \subset Y \setminus (\bar{S} \cup \bar{G})$. Thus, it is impossible to require K_X to be ample by Proposition 6.6.(4). The conditions (C5) and (C6) on Z follow immediately from Figure 11. We define $B := S + G + F + J_1$. Note that N^\sim passes through the node of \bar{F}_1 .

We take the birational morphism $\varphi: M \rightarrow Z$ so that the total transforms of $B + J_2$ and \bar{G}^+ in M form a configuration of curves as in Figure 12. Then, $\rho(M) = \rho(Z) + 5 = 17$

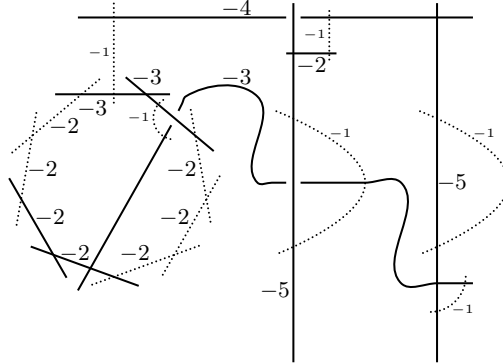


FIGURE 12. The rational surface M in Example 7.6

and $K_M^2 = -7$. Here, we have a disjoint union D of the following three linear chains of rational curves satisfying (C7):

$$D_1 = \text{LC}(4, 5, 3, 2, 2), \quad D_2 = \text{LC}(5, 2), \quad D_3 = \text{LC}(3, 3).$$

By Tables 1 and 2, D_1 , D_2 , and D_3 define toric singularities of type $T(1, 11, 3)$, $T(1, 3, 1)$, and $T(2, 2, 1)$, respectively, and

$$K_X^2 = K_M^2 + \delta(1, 11, 3) + \delta(1, 3, 1) + \delta(2, 2, 1) = -7 + 5 + 2 + 1 = 1.$$

Thus, (C8) and (C9) hold. The condition (C11) follows immediately from Figure 12. We shall prove (C10) by using Lemma 6.7. Then, it suffices to show $\Delta E > 1$ for the (-1) -curve E which joins the end (-4) -component of D_1 and the end (-2) -component of D_2 .

By Tables 1 and 2, we have

$$\Delta E = (1 - 3/11) + (1 - 2/3) > 1.$$

Hence, (C10) follows. We shall prove the simply connectedness of $M \setminus D$ in case $\mathbb{k} = \mathbb{C}$ by applying Lemma 6.10.1. We have $(d_1, d_2, d_3) = (1, 1, 2)$ and $(n_1, n_2, n_3) = (11, 3, 2)$ for the type $T(d_i, n_i, a_i)$ of the singularity defined by D_i for $1 \leq i \leq 3$. Looking at Figure 12, we have the following (-1) -curves:

- the (-1) -curve E which meets end components of D_1 and D_2 .
- a (-1) -curve meeting the end (-4) -component of D_1 and an end (-3) -component of D_3 .

Since $\gcd(d_1 n_1, d_2 n_2) = \gcd(11, 3) = 1$ and $\gcd(d_1 n_1, d_3 n_3) = \gcd(11, 4) = 1$, the conditions of Lemma 6.10.1 are all satisfied, and hence $M \setminus D$ is simply connected. Therefore, we have done all the tasks.

Example 7.7. Assume that $\text{char}(\mathbb{k}) \neq 3$ and we set $K^2 = 3$. We consider the same cubic pencil Φ as in Example 7.6. We also consider the same $\bar{F} = \bar{F}_1 + \bar{F}_2$, but define

$$\bar{S} := \bar{S}_1 + N^\sim \quad \text{and} \quad \bar{G} := \bar{G}^+ - L_1^\sim = \sum_{1 \leq i \neq 3 \leq 9} \Gamma_i.$$

Then, (C1), (C2), and (C3) hold automatically. Moreover (C4) by Lemma 6.3, since the dual graph of \bar{G} is A_8 and $\det(\bar{G}_i \bar{G}_j) \not\equiv 0 \pmod{\text{char}(\mathbb{k})}$. We have (A2) immediately from Figure 11. As in Example 7.6, (C5) and (C6) hold on Z . But, we define here G to be $S + F + J_1 + J_2$. Let $\varphi: M \rightarrow Z$ be the birational morphism such that the total transform B_M^+ of \bar{B}^+ is as in Figure 13. Then, $\rho(M) = \rho(Z) + 15 = 27$ and $K_M^2 = -17$. Here,

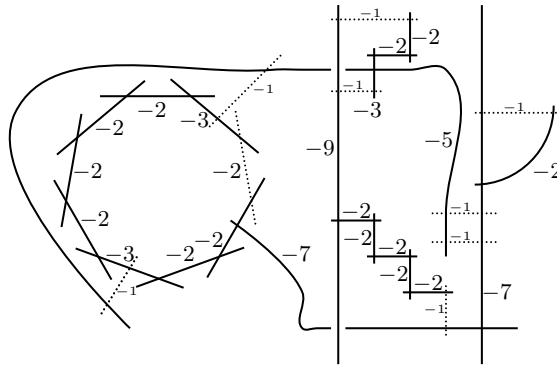


FIGURE 13. The rational surface M in Example 7.7

we have a disjoint union D of the following three linear chains of smooth rational curves satisfying (C7):

$$D_1 = (2, 7, 7, 2, 2, 3, 2, 2, 2, 2, 3) \quad D_2 = (5, 3, 2, 2), \quad D_3 = (9, 2, 2, 2, 2, 2).$$

By Tables 1 and 2, D_1 , D_2 , and D_3 define the toric singularities of type $T(1, 63, 34)$, $T(2, 4, 1)$, and $T(1, 7, 1)$, respectively, and

$$K_X^2 = K_M^2 + \delta(1, 63, 34) + \delta(2, 4, 1) + \delta(1, 7, 1) = -17 + 11 + 3 + 6 = 3.$$

Thus, (C8) and (C9) hold. The conditions (C11) and (A3) follow immediately from Figure 13. We shall prove (C10) by using Lemma 6.7. Then, it suffices to check $\Delta E_i > 0$ for $i = 1, 2, 3$ for the (-1) -curves E_1 , E_2 , and E_3 on M characterized by:

- E_1 joins the end (-2) -component of D_2 and the (-9) -curve in D_3 .
- E_2 joins the end (-2) -component of D_3 and the third component of D_1 which is a (-7) -curve.
- E_3 joins the end (-2) -component of D_1 and the second component of D_1 which is a (-7) -curve.

By Tables 1 and 2, we can calculate

$$\begin{aligned} \Delta E_1 &= (1 - 3/4) + (1 - 1/7) > 1, & \Delta E_2 &= (1 - 6/7) + (1 - 1/63) > 1, \\ \Delta E_3 &= (1 - 34/63) + (1 - 5/63) > 1. \end{aligned}$$

Hence, (C10) holds. The ampleness of K_X follows from (A2) and (A3) by Proposition 6.6.(4). We shall show that $M \setminus D$ is simply connected when $\mathbb{k} = \mathbb{C}$ by applying Lemma 6.10.1. We have $(d_1, d_2, d_3) = (1, 2, 1)$ and $(n_1, n_2, n_3) = (63, 4, 7)$ for the type $T(d_i, n_i, a_i)$ of the singularity defined by D_i for $1 \leq i \leq 3$. Looking at Figure 13, we have the following two (-1) -curves:

- the (-1) -curve E_1 which meets end components of D_2 and D_3 .
- a (-1) -curve meeting the end (-3) -component of D_1 and the end (-5) -component of D_2 .

Since $\gcd(d_2 n_2, d_3 n_3) = \gcd(8, 7) = 1$ and $\gcd(d_1 n_1, d_2 n_2) = \gcd(63, 8) = 1$, the conditions of Lemma 6.10.1 are all satisfied, and hence $M \setminus D$ is simply connected. Therefore, we have done all the tasks.

Example 7.8. Assume that $\text{char}(\mathbb{k}) \neq 3$ and we set $K^2 = 2$. We define

$$\phi_0 = \mathbf{x}^3 + \mathbf{yz}(\mathbf{y} + \mathbf{z}) \quad \text{and} \quad \phi_\infty = 3\mathbf{xyz}.$$

Then, Φ_0 is a smooth cubic curve, and $\Phi_\infty = L_1 + L_2 + L_3$ for the lines $L_1 = (\mathbf{x})_0$, $L_2 = (\mathbf{y})_0$, and $L_3 = (\mathbf{z})_0$. In particular, (C1) holds. Here, note that $P_3 = (0 : 0 : 1) = L_1 \cap L_2$ and $P_2 = (0 : 1 : 0) = L_1 \cap L_3$ are inflection points of Φ_0 , and that L_2 and L_3 are the tangent lines of Φ_0 at P_3 and P_2 , respectively (cf. Figure 14). The condition (C2) also holds. In fact, for $c \neq 0$, the divisor $\Phi_c = (\phi_0 + c\phi_\infty)_0$ is singular if and only if $c = -\omega^i$ for $i = 0, 1, 2$, where ω is a primitive cubic root of 1, and $\Phi_{-\omega^i}$ is a nodal rational curve with the

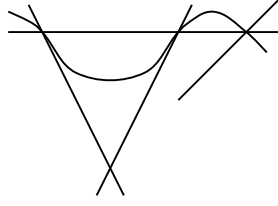
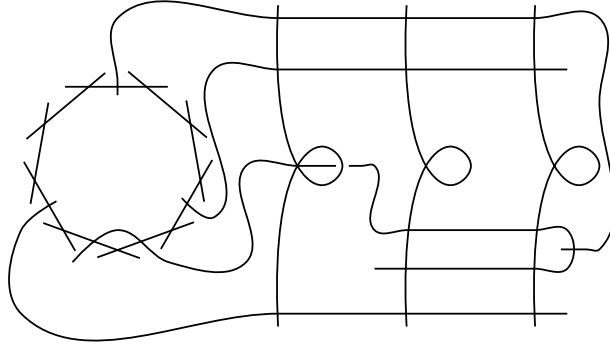


FIGURE 14. A pencil of cubics

node at $(1 : \omega^i : \omega^i)$. We define N to be the line passing through $(0 : 1 : -1) \in \Phi_0 \cap L_1$ and the node $(1 : 1 : 1)$ of Φ_{-1} . Thus, $N = (2x - y - z)_0$ (cf. Figure 14). On the minimal elliptic fibration $\pi: Y \rightarrow \mathbb{P}_{\mathbb{K}}^1$ defined by Φ , the configuration type of singular fibers is (I_9, I_1, I_1, I_1) , and the (-1) -curves exceptional for $Y \rightarrow \mathbb{P}_{\mathbb{K}}^2$ are mutually disjoint sections \bar{S}_1 , \bar{S}_2 , and \bar{S}_3 as in Figure 15. We may assume that \bar{S}_1 , \bar{S}_2 , \bar{S}_3 are contracted to the

FIGURE 15. The rational elliptic surface Y in Example 7.8

points $(0 : 1 : -1) \in \Phi_0 \cap L_1$, $P_3 = (0 : 0 : 1)$, and $P_2 = (0 : 1 : 0)$, respectively. The three singular fibers of type I_1 of π are the proper transforms of $\Phi_{-\omega^i}$ for $i = 0, 1, 2$. We set \bar{F}_1 and \bar{F}_2 to be the proper transforms of Φ_{-1} and $\Phi_{-\omega}$, respectively, and define $\bar{F} = \bar{F}_1 + \bar{F}_2$. The irreducible components of the singular fiber of type I_9 are labeled as $\Gamma_0 + \Gamma_1 + \cdots + \Gamma_8$ as a cyclic chain of rational curves in such a way that $\Gamma_0 = L_1^\sim$, $\Gamma_4 = L_2^\sim$, and $\Gamma_5 = L_3^\sim$, where L_i^\sim is the proper transform of L_i in Y for $i = 1, 2, 3$. Then,

$$\bar{S}_1\Gamma_0 = \bar{S}_2\Gamma_3 = \bar{S}_2\Gamma_6 = 1, \quad \text{and} \quad \bar{S}_j\Gamma_i = 0$$

for other (i, j) with $0 \leq i \leq 8$ and $1 \leq j \leq 3$. The proper transform N^\sim of N in Y is a bisection of π with self-intersection number zero passing through the node of \bar{F}_1 but no nodes of the other singular fibers of type I_1 . We have

$$N^\sim\bar{S}_1 = N^\sim\Gamma_4 = N^\sim\Gamma_5 = 1, \quad \text{and} \quad N^\sim\bar{S}_j = N^\sim\Gamma_i = 0$$

for $0 \leq i \neq 4, 5 \leq 8$ and $j = 2, 3$ (cf. Figure 15). We define $\bar{S} := \bar{S}_3 + N^\sim$. Since the union \bar{G}^+ of all the (-2) -curves on Y is just the singular fiber of type I_9 , we have (C3).

We define

$$\bar{G} := \bar{G}^+ - \Gamma_2 - \Gamma_3 - \Gamma_4 = \sum_{0 \leq i \neq 2,3,4 \leq 8} \Gamma_i.$$

We shall show (C4) without using Lemma 6.3. Assume that

$$a_1 \bar{S}_3 + a_2 N^\sim + \sum_{i=0}^8 m_i \Gamma_i \sim pH$$

for a Cartier divisor H , integers $a_1, a_2, m_0, \dots, m_8$ with $m_2 = m_3 = m_4 = 0$, where $p = \text{char}(\mathbb{k})$. Considering the intersection numbers with $\bar{S}_1, \bar{S}_3, N^\sim, \Gamma_1, \Gamma_2, \Gamma_5$, we have

$$a_2 + m_0 \equiv -a_1 + m_6 \equiv m_5 \equiv m_0 - 2m_1 \equiv m_1 \equiv a_2 - 2m_5 + m_6 \equiv 0 \pmod{p}.$$

Thus, $a_1 \equiv a_2 \equiv m_i \equiv 0 \pmod{p}$ for $0 \leq i \leq 6$. Moreover, we have $m_7 \equiv m_8 \equiv 0 \pmod{p}$ by considering the intersection numbers with Γ_0 and Γ_8 . Hence, (C4) holds. The condition (A2) fails. In fact, $\Gamma_4 \subset Y \setminus (\bar{S} \cup \bar{G})$. Thus, it is impossible to require K_X to be ample by Proposition 6.6.(4). The conditions (C5) and (C6) on Z follow from Figure 15 immediately. We define $B := S + G + F + J_1$.

We take the birational morphism $\varphi: M \rightarrow Z$ so that the total transforms of $B + J_2$ and \bar{G}^+ in M form a configuration of curves as in Figure 16. Then, $\rho(M) = \rho(Z) + 8 = 20$

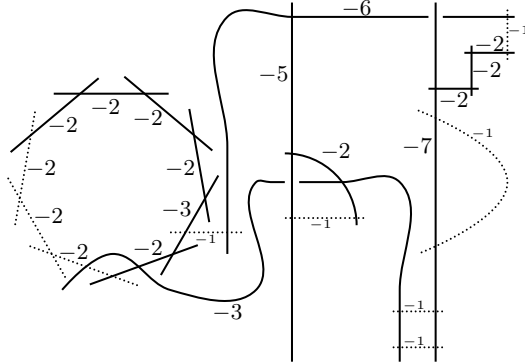


FIGURE 16. The rational surface M in Example 7.8

and $K_M^2 = -10$. Here, we have a disjoint union D of the following two linear chains of smooth rational curves satisfying (C7):

$$D_1 = \text{LC}(6, 5, 2, 3, 2, 3, 2, 2, 2, 2), \quad D_2 = \text{LC}(7, 2, 2, 2).$$

By Tables 1 and 2, D_1 and D_2 define toric singularities of type $T(3, 23, 4)$ and $T(1, 5, 1)$, respectively, and

$$K_X^2 = K_M^2 + \delta(3, 23, 4) + \delta(1, 5, 1) = -10 + 8 + 4 = 2.$$

Thus, (C8) and (C9) hold. The condition (C11) follows immediately from Figure 16. We shall prove (C10) using Lemma 6.7. Then, it suffices to show $\Delta E > 1$ for the (-1) -curve

E which joins the (-6) -curve of D_1 and the end (-2) -component of D_2 . By Tables 1 and 2, we have

$$\Delta E = (1 - 4/23) + (1 - 4/5) > 1.$$

Hence, (C10) holds. We shall prove the simply connectedness of $M \setminus D$ in case $\mathbb{k} = \mathbb{C}$ by applying Lemma 6.10.1. We have $(d_1, d_2) = (3, 1)$ and $(n_1, n_2) = (23, 5)$ for the type $T(d_i, n_i, a_i)$ of the singularity defined by D_i for $i = 1, 2$. Since the (-1) -curve E meets end components of D_1 and D_2 and since $\gcd(d_1 n_1, d_2 n_2) = \gcd(3 \times 23, 5) = 1$, the conditions of Lemma 6.10.1 are satisfied, and hence $M \setminus D$ is simply connected. Therefore, we have done all the tasks.

Finally, we shall prove our main result. We restate the result.

Main Theorem. *For any algebraically closed field \mathbb{k} and for any integer $1 \leq K^2 \leq 4$, there exists an algebraically simply connected minimal surface \mathbb{S} of general type over \mathbb{k} with $p_g(\mathbb{S}) = q(\mathbb{S}) = \dim H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = 0$ and $K_{\mathbb{S}}^2 = K^2$ except $(\text{char}(\mathbb{k}), K^2) = (2, 4)$, where $\Theta_{\mathbb{S}/\mathbb{k}}$ denotes the tangent sheaf. Moreover, one can find such a surface with ample canonical divisor when $1 \leq K^2 \leq 4$, except $(\text{char}(\mathbb{k}), K^2) = (2, 1), (2, 2)$, and $(2, 4)$.*

Proof. Assume that $(p, K^2) \neq (2, 4)$. Then, we have an algebraically simply connected minimal surface \mathbb{S} of general type defined over \mathbb{k} such that $p_g(\mathbb{S}) = q(\mathbb{S}) = \dim H^2(\mathbb{S}, \Theta_{\mathbb{S}/\mathbb{k}}) = 0$ and $K_{\mathbb{S}}^2 = K^2$ by Propositions 6.8 and 6.11 applied to Examples 7.1–7.8 above. In fact, the case $K^2 = 1$ is treated in Examples 7.3 (when $p \neq 2$) and 7.6 (when $p \neq 3$); the case $K^2 = 2$ in Examples 7.1 (when $p \neq 2, 3$), 7.2 (when $p \neq 2$), and 7.8 (when $p \neq 3$); the case $K^2 = 3$ in Examples 7.4 (when $p \neq 2$) and 7.7 (when $p \neq 3$); and the case $K^2 = 4$ in Example 7.5 (when $p \neq 2$). Here, K_X is not ample only in Examples 7.6 and 7.8. Thus, we can require $K_{\mathbb{S}}$ to be ample if $(p, K^2) \neq (2, 1), (2, 2)$. Hence, the proof has been completed. \square

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(Yongnam Lee) DEPARTMENT OF MATHEMATICS
 SOGANG UNIVERSITY, SINSU-DONG, MAPO-GU, SEOUL 121-742 KOREA
E-mail address: ynlee@sogang.ac.kr

(Noboru Nakayama) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
 KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
E-mail address: nakayama@kurims.kyoto-u.ac.jp

